Applied Economics and Finance; Vol. 12, No. 2; 2025
ISSN 2332-7294 E-ISSN 2332-7308
Published by Redfame Publishing

Multi-Cut Implementation of Hub-and-Spoke Problem with Congestion Consideration

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Received: October 1, 2024 Revised: October 30, 2024 Accepted: April 1, 2025

Available online: April 25, 2025 URL: https://doi.org/10.11114/aef.v12i2.7667

Abstract

In this study, we examine a p-hub location problem where the objective function encompasses regular transportation costs, fixed expenses, and congestion costs associated with hubs. We demonstrate that the model for this problem is a convex mixed integer programming problem. To solve the problem, we introduce a multi-cut cutting-plane method and compare its performance to the existing single cut version method. Our findings show that the multi-cut method outperforms the single-cut method in terms of the time required to reach a solution. The results of numerical experiments conducted to support this comparison are also presented.

Keywords: Cutting-plane, Single-cut, Multi-cut, congestion, hub location

1. Introduction

In recent years, the hub location problem (HLP) has emerged as a focal point of research and practical applications. Its prominence stems from the potential to optimize transportation logistics by consolidating products at strategic hubs before final distribution, thereby mitigating transportation costs. Various types of HLP models have been developed to address diverse real-world transportation challenges. Ghodratnama et al. classify HLP into four main categories, each targeting specific optimization objectives:

1. **Hub Covering Problems**: These seek to balance meeting demand while minimizing the costs associated with opening new hubs.
2. **Fixed Cost Hub Location Problems:** The focus here lies in determining the optimal number of hubs, considering fixed costs.
3. **Median Hub Allocation Problems:** These aim to streamline transportation costs by strategically locating hubs.
4. **Hub Center Location-Allocation Problems:** The primary goal is to minimize the

   maximum transportation cost between nodes, optimizing network efficiency.

In exploring prior research on HLP, O'Kelly conducted one of the earliest investigations into the field. Their study centered on a HLP variant where fixed costs are incurred for hub establishment, with the number of hubs serving as a decision variable. O'Kelly expanded on this work by highlighting the influence of flow scale economies on HLP decisions, underscoring the significance of considering such factors. Alumur and Kara provided an extensive review of HLP, offering a comprehensive classification of hub location models based on various objective functions, constraints, and demand patterns. Addressing delivery uncertainties, Mohammadi et al. investigated the single-allocation p-HLP, proposing a meta-heuristic algorithm to address uncertainty in deliveries.

Azizi et al. advocated for the integration of backup hubs for demand nodes to mitigate disruptions in hub operations. By strategically incorporating backup infrastructure, the potential costs of operational disruptions can be significantly reduced, ensuring the continuity of logistics operations. In a similar vein, Rostami et al. proposed a single-hub allocation model that integrates backup hubs to bolster operational reliability. By augmenting existing hub infrastructure with backup facilities, the model enhances the robustness of hub operations, minimizing the impact of potential disruptions.

On a broader scale, Soylu et al. introduced a bi-objective multiple allocation phub median model that considers both transportation expenses and consumer satisfaction metrics. This holistic approach ensures that optimization efforts not only minimize operational costs but also prioritize customer satisfaction, reflecting a nuanced understanding of the multifaceted nature of hub management. In a complementary study, Karimi et al. addressed a single-allocation HLP with a multi-objective model that optimizes both transportation costs and travel time. By simultaneously considering these key performances metrics, the model offers a comprehensive framework for hub optimization, balancing cost-efficiency with operational effectiveness.

In terms of solution strategies, Abdinnour et al. explored the efficacy of simulated annealing as an AI heuristic for solving the p-Hub Median Problem. Their comparative analysis highlights the potential of AI-driven approaches in addressing complex optimization challenges inherent in hub management. Mokhtar et al. focused on the uncapacitated p-hub median problem, employing Benders decomposition and proposing efficiency-enhancing strategies. By leveraging advanced mathematical techniques, their study contributes to the development of efficient solution methodologies for hub optimization.

The Uncapacitated HLP has been formulated as a quadratic integer programming problem by Abdinnour et al. Their study presents a branch-and-bound solution approach along with a Genetic algorithm for solving the model, contributing to the development of efficient solution methodologies for this challenging optimization problem. Addressing the critical aspect of hub capacity, Boffey et al. provided a comprehensive review on hub capacity considerations and their implications, shedding light on key factors influencing hub operations.

While congestion in HLP has been a relatively understudied aspect, Parajuli et al. attempted to model congestion costs arising from major disruptions using a convex function. Their approach, coupled with an enumeration algorithm, suggests the implementation of a decentralized protection system in the event of congestion, offering insights into effective congestion management strategies. Najy et al. addressed the uncapacitated HLP incorporating flow-dependent economies of scale and congestion considerations into the multiple-allocation version of the problem. They proposed a specific Benders decomposition strategy to solve the problem, contributing to the advancement of solution methodologies for complex HLP variants.

In a nonlinear mixed integer programming model with congestion costs, De et al. employed the Benders Decomposition solution approach to strike a balance between transportation and congestion costs. Their study underscores the importance of considering congestion dynamics in hub location decisions, offering valuable insights for effective logistics management. Azizi et al. developed a mixed integer programming model with a nonlinear congestion penalty in the objective function. Leveraging a cutting plane approach under the single allocation assumption, their study employs linearization techniques to efficiently solve the problem, contributing to the arsenal of solution methodologies for congestion-aware hub location optimization.

Including congestion costs is crucial across various industries, significantly impacting customer satisfaction, such as in the airline industry with connecting flights. Understanding how to effectively implement the cutting-plane algorithm when the model includes congestion is essential. However, existing studies, such as Azizi et al., primarily focus on multi-cut approaches without extensive examination of single-cut methods. This gap in literature prompts the need for a comparative analysis of single-cut versus multi-cut implementations.

In our research, we extend the work of Azizi et al. by implementing their algorithm in C language and comparing two implementation approaches: single-cut and multi-cut. The cutting-plane method used by Azizi et al. introduces multiple cuts per iteration, where each cut corresponds to a node in the network. Our study presents a single-cut implementation and compares its performance with the multi-cut approach. The results demonstrate the efficiency of the single-cut method in solving the hub location problem. The rest of the paper is organized as follows. The problem definition is presented in Section 3. The two algorithms are then introduced in Section 4. Finally, in Section 5, we offer numerical results, and in Section 6, we present the conclusion.

2. Problem Definition

Suppose that there is a set of nodes N, which can be divided into two sets of origins and destinations. A collection of hubs is chosen instead of linking all origins to all destinations. Before reaching its final destination, the flow travels from the origin to a hub assigned to that origin, then from that hub to another hub linked to the destination. The location of hubs and the connections between non-hub nodes and hubs are the subject of this problem. In addition, assume that the demand is random. The goal of the problem is to reduce fixed costs, transit costs, and congestion costs. To investigate the solution approaches, we first introduce the model presented in. Here are some notations associated with this model.

The decision variable, x_{ijkm} is a binary variable that equals 1 if the flow from node i to node j is routed through hubs located at nodes k and m; otherwise, it equals 0. The variable z_{ik} is also binary and indicates whether node i is allocated to hub k; a value of 1 signifies allocation, while 0 indicates no allocation. Finally, ykl is another binary variable that equals 1 if hub k is equipped with capacity level l; otherwise, it equals 0. The original model for this problem presented in is as follows:

$$\min f(x,y,z) + \frac{\theta}{2}\sum_{k\in N}\mathbb{E}[L_k(x,y)] \tag{1a}$$

$$s.t. \qquad \sum_{k\in N} z_{ik} = 1, \qquad \forall i \in \mathrm{N} \tag{1b}$$

$$z_{ik} \le z_{kk}, \qquad \forall i,k \in \mathrm{N} \tag{1c}$$

$$\sum_{k\in N} z_{kk} = p \tag{1d}$$

$$\sum_m x_{ijkm} = z_{ik}, \qquad \forall i,k,j \in \mathrm{N} \tag{1e}$$

$$\sum_k x_{ijkm} = z_{ik}, \quad \forall i,j,m \in \mathrm{N} \tag{1f}$$

$$\sum_i\sum_j\sum_m \lambda_{ij}x_{ijkm} = \sum_l \mu_{kl}y_{kl}, \qquad \forall k \in \mathrm{N} \tag{1g}$$

$$\sum_l y_{kl} = z_{kk}, \qquad \forall k \in \mathrm{N} \tag{1h}$$

$$x_{ijkm}, y_{kl}, z_{ik} \in \{0,1\}, \qquad \forall i,j,k,m \tag{1i}$$

where the objective function $f(x,y,z)$ is defined as:

$$f(x,y,z) = \sum_i\sum_j\sum_k\sum_m c_{ijkm}x_{ijkm} + \sum_k\sum_l F_{kl}y_{kl}$$

The objective function (1a) aims to minimize the total network cost, which includes transportation cost, hub fixed cost, and congestion costs. The function $f(x,y,z)$ comprises of two main parts: the first term represents the total transportation cost of the flow between all origin and destination nodes, and the second term calculates the cost of equipping hubs. The last term of the objective function incorporates the congestion cost of hubs. Constraint set (1b) ensures that every node is assigned to exactly one hub. Constraint (1c) guarantees that a node is assigned to an open hub. Constraint (1d) ensures that exactly p hubs are opened in the network. Constraints (1e) and (1f) ensure that the arcs being used are hub arcs, and the capacities of the hubs are respected. Constraint set (1g) demonstrates the capacity constraints at hubs, which can be interpreted as keeping the queue system stable. Constraint set (1h) guarantees that a hub is quipped if it is opened. In this model. The congestion is represented by the expected value of the total number of demands waiting in the hubs. Based on [5], when arrival follows a Poisson process, it can be calculated as follows:

$$\mathbb{E}[L_k(x,y)] = \frac{1}{2}\left\{\frac{(1+\sum_l c_{kl}^2 y_{kl})\sum_i\sum_j\sum_m \lambda_{ij}x_{ijkm}}{\sum_l \mu_{kl}y_{kl} - \sum_i\sum_j\sum_m \lambda_{ij}x_{ijkm}} + \frac{(1-\sum_l c_{kl}^2 y_{kl})\sum_i\sum_j\sum_m \lambda_{ij}x_{ijkm}}{\sum_l \mu_{kl}y_{kl}}\right\}$$

where, c_{kl} is squared coefficient of variation of service times of node k when it is equipped by capacity l. We have c_{kl}can be calculated as follows:

$$c_{kl}^2 = c\mu_{kl}^2\sigma_{kl}^2$$

So, because of the congestion cost structure, we have a nonlinear mixed-integer programming problem. [5] introduced the new variables w, v, R and ρ, where:

$$\rho_k = \frac{\sum_i\sum_j\sum_m \lambda_{ij}x_{ijkm}}{\sum_l \mu_{kl}y_{kl}},$$

and

$$R_k = \frac{\sum_i\sum_j\sum_m \lambda_{ij}x_{ijkm}}{\sum_l \mu_{kl}y_{kl} - \sum_i\sum_j\sum_m \lambda_{ij}x_{ijkm}} \qquad \forall k \in N,$$

and reformulated the model with linear objective function and nonlinear constraints as follows:

$$\min f(x,y,z) + \frac{\theta}{2}\sum_{k\in N}[R_k + \rho_k \sum_l c_{kl}^2 (v_{kl} - w_{kl})] \quad (3a)$$

$$\text{s.t.} \quad (1b) \text{ to } (1i) \quad (3b)$$

$$\sum_i \sum_j \sum_m \lambda_{ij} x_{ijkm} = \sum_l \mu_{kl} w_{kl} \quad (3c)$$

$$w_{kl} \le y_{kl}, \quad \forall k,l \in N \quad (3d)$$

$$\rho_k = \frac{R_k}{1+R_k}, \forall k,l \in N \quad (3e)$$

$$\sum_l \mu_{kl} w_{kl} = \rho_k, \forall k \in N \quad (3f)$$

$$\sum_l v_{kl} = R_k, \forall k \in N \quad (3g)$$

$$\sum_l y_{kl} \le 1, \forall k \in N \quad (3h)$$

$$v_{kl} - M y_{kl} \le 0, \forall k \in N \quad (3i)$$

$$R_k - \sum_l v_{kl} = 0, \forall k \in N \quad (3j)$$

$$x_{ijkm}, y_{kl}, z_{ik} \in \{0,1\}, \quad \forall i,j,k,m,l \in N \quad (3k)$$

$$0 \le \rho_k \le 1, 0 \le w_{kl} \le 1, \forall k,l \in N \quad (3l)$$

$$R_k, v_{kl} \ge 0, \forall k,l \in N \quad (3m)$$

Because of the nature of the congestion cost, assuming $c_{kl} \ge 1$, $\forall k,l \in N$ in equation (2), the coefficient of variables ρ are non-positive. So that the model tries to make them larger as much as possible, and we can transform (3e) as follows:

$$\rho_k = \frac{R_k}{1+R_k}, \forall k,l \in N.$$

To verify the convexity of the problem, we calculated the Hessian matrix associate with each of the nonlinear constraints as follows:

$$M = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{(1+R)^3} \end{bmatrix}.$$

The matrix M has nonnegative eigen values and so is positive semi-definite. Thus, it is now verified that this is a convex programming problem. To linearize the model, [5] approximated the concave function in the right-hand side of constraint (??) by a set of upper-bounding affine functions and replaced the one $k \in N$ by a set of constraints as follows:

$$\rho_k - \left(\frac{R_k^h}{1+R_k^h} + \frac{-1}{(1+R_k^h)^2}\left(R_k - R_k^h\right)\right) \le 0, \forall k,l,R_k^h \in P_k$$

where P_k is a collection set of R_k^h values. Because the upper bounds approximate the right-hand side of a set of constraints, the approximated problem is a relaxed version of the preceding one, and its best value is a lower bound.

Algorithm 1 Single-Cut Algorithm

1: **Input**: $Z^{UB}, Z^{LB}, h = 0, H = \emptyset, \epsilon$.

2: **while** $Z^{UB} - Z^{LB} \ge \epsilon$ **do**

3: To get the optimal values for variables k, solve the master problem and place them

in R_k.

4: Build a cut as follows:

$$\sum_{k-1}^{K} \rho_k \le \sum_{k-1}^{K} \left[\frac{1}{\left(1+R_k^h\right)^2} R_k + \frac{\left(R_k^h\right)^2}{\left(1+R_k^h\right)^2} \right] \tag{4}$$

5: Add the cuts to the master problem and set H.

6: $h = h + 1$.

7: **End**

3. Algorithm

In this section, we have presented a single-cut version algorithm to solve the linearized model. Here, we go through the algorithm step by step. This algorithm requires an upper bound for problem (3). The other inputs are the acceptable optimality gap ϵ, the collection set of cuts H, the iteration counter h. At the beginning of iteration h, we set up and solve a master problem by modifying problem (3) replacing the constraint set (3e) with the set of approximating linear constraints in H. When we obtained the optimal solution $(x_{ijkm}^{*h}, y_{kl}^{*h}, z_{ik}^{*h}, \rho_k^{*h}, R_k^{*h}, w_{kl}^{*h}, v_{kl}^{*h})$ the set H is then updated by adding the new cut obtained by entering R_k^{*h} in (4). Next, we update the iteration counter h and go to the next iteration. In contrast to our work, the existing implementation of the algorithm presented by adds n many cuts of every single ρ_k, $k \in N$ in each iteration. A comparison of these two methods is presented in the following section, which is based on the implementation of the two algorithms for a collection of problem instances.

4. Numerical Results

To implement the algorithm, we first wrote the model in AMPL and produced some instances in MPS format. The algorithm code is written in C language in Visual Studio 2019 and Gurobi solver is used to solve the master problems in iterations. The objective function value and CPU times are mentioned in the following table. The parameters N, P and L show the number of nodes, number of hubs and the number of possible capacities respectively. The code is run for 100 iterations for both algorithms. Here, we have a comparison of the two algorithm's implementation.

Alg.	N	P	L	Upper Bound	CPU
SingleCut.	6	2	7	699.88	66.79s
	7	3	7	954.25	1151.39s
	8	2	8	1154.68	706.68s
	10	3	7	1751.41	10182.31s
MultiCut.	6	2	7	722.99	58.21s
	7	3	7	1035.73	920.75s
	8	2	8	1165.85	1175.61s
	10	3	7	1894.33	20142s

5. Conclusion and Future work

The accurate method meets difficulty as the size of the problem grows larger, according to numerical results. The single-cut version provided in this work appears to outperform the multi-cut version presented by [5] for the same number of iterations, as it delivers roughly the same quality of solutions in a much shorter time. This conclusion is acceptable because increasing the number of cuts in each iteration results in curse of dimensionality. For future efforts, we propose using a heuristic algorithm before stating the iterations. This can increase the algorithm's performance since cut production will be based on better points, and it will reduce the effort required in the initial iterations to determine the correct area for which affine functions must be built.

Acknowledgments

I would like to express my sincere gratitude to the Department of Operations Research and Engineering Management at Southern Methodist University for providing the computational resources necessary for this research. Special thanks to my colleagues who provided valuable insights during the development and implementation of the algorithms. I also appreciate the constructive feedback from the anonymous reviewers that helped improve the quality of this manuscript.

Authors' contributions

Niloofar Fadavi was responsible for the conceptualization, methodology, formal analysis, investigation, and writing the original draft. Yasin Ghahremanloo contributed to validation, and data curation and revision. Sina Akbari assisted with manuscript revision. All authors read and approved the final manuscript.

Funding

Not applicable.

Competing interests

The author declares that there are no competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Informed consent

Not applicable.

Ethics approval

The Publication Ethics Committee of the Redfame Publishing.

The journal's policies adhere to the Core Practices established by the Committee on Publication Ethics (COPE).

Provenance and peer review

Not commissioned; externally double-blind peer reviewed.

Data availability statement

The data that support the findings of this study, including the problem instances and algorithm implementations, are available on request from the corresponding author.

Data sharing statement

No additional data are available.

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