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Dynamic Price Dispersion of Seasonal Goods in Bertrand–Edgeworth Competition

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Abstract

The primary distinguishing characteristics of various revenue management systems include fixed capacities, uniform products, and customers' sensitivity to pricing when making purchase decisions. Even with only two or three competitors, the competition remains intense. This study examines the sub-game perfect Nash equilibrium of price competition within an oligopoly market where goods are perishable assets. Each seller possesses a single unit of a non-replenishable good and competes by setting prices to sell their inventory over a finite sales period. Each period, buyers seek to purchase one unit of the good, and the number of buyers entering is random. All sellers' prices are visible to buyers, and there is no cost associated with searching for goods. Through stochastic dynamic programming methods, sellers' optimal responses can be determined by treating the competition as a one-time price-setting game, considering the remaining sales periods and the current demand pattern. Assuming a binary demand model, it is demonstrated that the duopoly model exhibits a unique Nash equilibrium. In addition, it is shown that the oligopoly model does not result in price dispersion based on a discussed metric. Moreover, when a generalized demand model is considered, the duopoly model features a unique Nash equilibrium with mixed strategies, whereas the oligopoly model exhibits a unique symmetric Nash equilibrium with mixed strategies.

Keywords: price dispersion, mixed strategy, sub-game perfect, Nash equilibrium, dynamic programming, stochastic programming, duopoly, oligopoly

1. Introduction

The aim of revenue management is to effectively control the sale of a limited quantity of resources (like hotel rooms, airline seats, advertising slots, etc.) to potential customers (the market). Airlines represent one instance of businesses managing perishable capacity, where they competitively price fixed capacities for sale within a constrained sales timeframe. Online travel agencies like Expedia facilitate this competition by aggregating and displaying real-time flight prices from various airlines, enabling customers to compare product quality and pricing options. In today's rapidly evolving technological landscape, corporate managers are increasingly savvy, with revenue management playing a pivotal role in their decision-making processes. This is because, in a competitive market, having insight into competitors' potential actions is crucial for making informed decisions. For instance, companies might miss out on sales opportunities if they set prices too high, unaware of their competitors' likely pricing strategies. Consequently, there has been significant recent interest and research dedicated to understanding competition in revenue management. In this paper, we delve into an oligopoly stochastic model where sellers compete by adjusting prices over multiple periods. Building upon the Bertrand-Edgworth competition framework, this advanced model seeks to elucidate managers' considerations and pricing behaviors under uncertainty.

The paper's organization is as follows: Section 2 provides a review of relevant literature. In section 3, we define the problem and provide equilibrium properties for different demand scenarios. Section 4 details the results obtained from numerical experiments. Lastly, section 5 concludes the study by discussing findings and suggesting directions for future research. The theoretical underpinnings for these properties are provided in the appendix.

2. Related Literature

In this section, we've organized the related works into two distinct categories. We begin with a discussion of static competition models, followed by a concise introduction to dynamic competition models.

2.1 Static Models

The two main models of oligopoly are the Cournot model, where companies compete on how much they produce (Cournot, 1838), and the Bertrand model, where they compete on prices (Bertrand, 1988). In the Cournot model, companies decide how much to produce, and then the market decides the prices. In the Bertrand model, at the start of each period, companies set their prices, assuming they can meet all the demand for their product. In this model, each company sets its price based on its production costs. (Edgeworth, 1925) demonstrated that when considering capacity constraints, the Bertrand model lacks a pure strategy Nash equilibrium. This study delves into the Bertrand-Edgeworth model, which incorporates capacity limitations into Bertrand's competition. In this situation, customers typically select the company with the lowest price, and if two companies offer similar prices, customers randomly decide between them. The random order of customer arrivals introduces demand uncertainty. Although a pure-strategy equilibrium may not be guaranteed in this context, mixed-strategy equilibria are observed. See (Allen & Hellwig, 1986), (Dasgupta & Maskin, 1986), (Levitan & Shubik, 1972). Exploring the Bertrand-Edgeworth model provides valuable insights into revenue management competition. However, the lack of a pure-strategy equilibrium complicates the analysis and prediction of market dynamics. While the model offers simplicity, it may not provide precise insights into market conditions, especially for more complex scenarios involving multiple customer segments. Consequently, other relevant studies have explored mixed-strategy Nash equilibrium alongside the development of these fundamental models.

2.2 Dynamic Models

After the emergence of several static models, many researchers have turned their attention to dynamic models of competition. One of the pioneers in this domain was (Chamberlin, 1933), who was among the earliest to emphasize that repeated interactions among rivals in an oligopoly market could enable collusion. With frequent interaction, businesses can use the threat of price wars to dissuade rivals from engaging in collusive pricing strategies. Another noteworthy contribution is Dudy's paper, which explores dynamic competitions with capacity constraints (Dudey, 1992). Although not present in the Bertrand-Edgeworth competition, Dudy demonstrated that the subgame perfect equilibrium is unique in dynamic competition. Establishing this equilibrium's uniqueness was a significant achievement because the lack of uniqueness can pose challenges in economic applications. Without a unique solution, it becomes difficult to ascertain which equilibrium each competitor considers, impeding legitimate comparative static exercises (Osborne & Pitchik, 1986). Dudy's model involves companies setting prices in each period to maximize their undiscounted profits within a duopoly variant of Bertrand-Edgeworth competition with capacity constraints. They assume a deterministic demand structure in their model. (Biglaiser & Vettas, 2004) investigate a dynamic price game where businesses face limited capacities and customers are strategic. They find that there is no pure-strategy subgame perfect equilibrium and that the seller's market shares are likely to be asymmetric. (Anderson & Schneider, 2007) examine the impact of search costs on equilibrium in a dynamic price model. The study shows that when search costs are not zero, duopolistic prices exceed those of monopolists. (Mantin, 2008) analyzes a multiperiod duopoly pricing game involving a homogeneous perishable good sold to consumers who visit one of the retailers in each period. (Perakis & Sood, 2004) study an open-loop equilibrium, where prices are set at the start of the horizon and remain unchanged until the end. Therefore, they do not require subgame perfection. They utilize variational inequality tools and conduct comparative statics on equilibrium prices as a function of demand-capacity ratios. (Deneckere & Peck, 2012) derive dynamic price dispersion in perfect competition by developing a dynamic version of Prescott's hotel model. Their model integrates price variation across periods, adjusting the demand function based on past revealed demand. Companies have the option to set a low price for guaranteed sales or a higher price for sales only in high-demand periods. Future decisions are made based on updated demand, without considering discount factors. (Martínez-de Albéniz & Talluri, 2011) investigated perfect competition of a homogeneous product when the demand function follows a Bernoulli random variable. They derived a mathematical solution for equilibrium price paths and demonstrated a structural property of the equilibrium policy, where the seller with the lower equilibrium reservation value sells a unit at a price equal to the competitor's equilibrium reservation value. In (Gallego & Hu, 2014), the analysis initially focuses on a deterministic demand function and later demonstrates that, under certain conditions, the results can approximate scenarios where the demand function is random. (Sun, 2017) examined oligopolistic competition where each firm has one unit to sell, and products are non-perishable. They identified equilibrium price dispersion, particularly proving its uniqueness when demand follows a Bernoulli distribution.

2.3 Contributions

In recent years, there has been a surge of research aimed at understanding and improving customer behavior. However, comparatively less attention has been given to modeling competition dynamics. This discrepancy can be attributed to the inherent challenge of establishing the existence of subgame perfect equilibrium in stochastic revenue management games, even in relatively straightforward models such as the Bertrand-Edgeworth model, which lacks a single pure Nash equilibrium. Furthermore, much of the existing research has focused on static models and infinite games, often overlooking the presence of mixed strategies or resorting to special cases to find pure Nash equilibrium. Our research seeks to fill this gap by addressing equilibrium dynamics within a finite long-run horizon. We specifically focus on establishing the existence of a unique mixed strategy Nash equilibrium in a duopoly model with generalized demand. Additionally, we demonstrate the uniqueness of symmetric mixed strategy Nash equilibrium in an oligopoly model. Our objective is to conduct a comprehensive mathematical analysis to understand how various factors influence market dynamics and sellers' behaviors in equilibrium. While meeting all model assumptions may pose challenges, our model serves as a valuable benchmark for exploring the effects of different parameters on market behavior and price dispersion characteristics. To accomplish this, we build upon the mathematical framework introduced by Sun (2017).

3. Methodology

In this section, we delve into the dynamics of the Bertrand-Edgeworth competition, starting with a focus on the Bernoulli demand distribution and then expanding our analysis to cover general demand distributions. In each demand scenario, we explore the conditions for equilibrium in both duopoly and oligopoly competitions. Before delving into the intricacies of equilibrium analysis, let's establish the basics of Nash equilibrium. Imagine a strategic game involving N players, where each player has a set of pure strategies. A mixed strategy in this context involves randomization over this set. It's assumed that the mixed strategies adopted by the competitors are independent. This leads us to a new version of the game where mixed strategies replace the simpler pure strategies, known as the mixed extension of the original game. According to (Osborne & Rubinstein, 1994), we define a mixed strategy Nash equilibrium of a strategic game as a Nash equilibrium of its mixed extension. Another important concept is the subgame perfect Nash equilibrium (SPNE), which applies to dynamic games. A set of strategies exhibits an SPNE if, at every point of the dynamic game, which constitutes a subgame, the outcome corresponds to a Nash equilibrium. See (Baye & Morgan, 2004) for more details.

3.1 Mathematical Model

We're examining a market with N competing firms, each offering a single unit of a uniform, perishable product. Time is divided into discrete intervals labeled from 1 to T, representing a finite time period. Prior to knowing the demand for each period, firms must determine their prices. Customers will only make a purchase if the price is lower than their predetermined reservation price, which remains consistent across all customers. Upon arriving at the market each period, customers seek to buy one unit and leave if they cannot make a purchase. In instances where two firms price their products the same, they have an equal probability of making a sale. The objective for each firm is to maximize the present value of its profits, taking into account any discounts. Let's introduce some key terms for further exploration:

- T: Represents the remaining number of time units until the product expires.
- $\bar{p}$: Denotes the reservation price.
- $V(\bar{p}, N, T)$: Represents the anticipated option-value for a firm when there are N competitors in the market and T periods remain.
- $P^*_{N,T}$: Indicates the minimum price each firm tends to consider when there are N competitors in the market and T periods remain.
- δ: Represents the discount factor.

Determining option-values plays a pivotal role in crafting a company's pricing strategy. This value represents the expected profit a company would earn by maintaining the price at the reservation level until the planning horizon ends. By possessing this option, companies establish a minimum level for the profit achievable through their chosen pricing strategy. Hence, according to the definition of option-value, we arrive at the following equation:

$$V(\bar{p}, T, N) = 0, \tag{1}$$

for $N > T$. In the next sections, we explore Nash equilibrium for the duopoly model, then create and find the equilibrium for the oligopoly market.

3.2 Binary Demand

In this section, we delve into the market dynamics when the demand distribution follows a Bernoulli distribution. Initially, we discuss certain properties of option-value applicable to both duopoly and oligopoly markets. Subsequently, we explore the equilibrium for each scenario individually. Let's introduce additional notations:

- q : The probability of zero demand.
- $\Gamma(N, q, T)$: Represents a scenario where N firms operate within a market over a planning period of T, with the demand parameter set as q.

We can perform the computations for our case using the same reasoning as (Sun, 2017). For this market, the closed-form of the option-value can be calculated. Based on Bayes' Theorem, we have:

$$\mathrm{V}(\bar{p}, \mathrm{N}, \mathrm{T}) = q\delta V(\bar{p}, N, T-1) + (1-q)\delta V(\bar{p}, N-1, T-1). \tag{2}$$

Similarly, $V(\bar{p}, 1, T)$ can be computed as follows:

$$V(\bar{p}, 1, T) = q\delta V(\bar{p}, 1, T-1) + (1-q)\bar{p} = (1-q)\bar{p} + q\delta(1-q)\bar{p} + \cdots + (q\delta)^{T-1}(1-q)\bar{p} = (1-q)\bar{p}(1-(q\delta)^T)/(1-q\delta). \quad (3)$$

According to (2) and (3), we can calculate $V(\bar{p}, N, T)$ according to the following equation:

$$V(\bar{p}, N, T) = \sum_{i=1}^{T-N+1} [(q\delta)^{i-1}(1-q)\delta V(\bar{p}, N-1, T-i)], \quad (4)$$

when $N \leq T$. Due to the presence of an option-value, firms set a minimum threshold for the price they're willing to accept, within a specific period. To determine this threshold, we seek a price denoted as $P^*_{N,T}$, where the anticipated profit from setting this price aligns with the current option-value. This results in the following equation being satisfied:

$$V(\bar{p}, N, T) = q\delta V(\bar{p}, N, T-1) + (1-q)P^*_{N,T}, \quad (5)$$

for $N \leq T$. In conclusion, having $V(\bar{p}, N, T)$, we can calculate $P^*_{N,T}$ as:

$$P^*_{N,T} = \frac{V(\bar{p}, N, T) - q\delta V(\bar{p}, N, T-1)}{1-q} = \delta V(\bar{p}, N-1, T-1), \quad (6)$$

for $N \leq T$. Equation (6) retains a straightforward interpretation. As highlighted in (Sun, 2017) concerning non-perishable assets, this concept remains relevant in our context. With a Bernoulli demand distribution, at most, only one buyer enters the market in each period. Thus, $P^*_{N,T}$ is the price at which a seller is indifferent between selling or letting others sell when there is a one-unit demand. The profit in the former case is $P^*_{N,T}$, while the profit in the latter case is $\delta V(\bar{p}, N-1, T-1)$. Consequently, $P^*_{N,T} = \delta V(\bar{p}, N-1, T-1)$.

3.3 Duopoly Competition with Binary Demand

In this section, we examine the equilibrium for duopoly markets. To examine the mixed-strategy equilibrium, we model the sellers' strategy as a random variable. Here, we introduce some notations:

- $F_{iT}(p)$: the probability that the seller i sets a price less than or equal to p in period T,
- $F_{-iT}(p)$: the joint probability that all sellers except i set prices less than or equal to p in period T,
- $F_{iT}(p_-) = \lim_{p \to p^-} F_{iT}(p)$,
- $I_{iT} = \sup\{p \mid F_{iT}(p) = 0\}$,
- $U_{iT} = \inf\{p \mid F_{iT}(p) = 1\}$.

Lemma 3.1 and 3.2 demonstrate the relationship between the minimum acceptable price $P^*_{N,T}$ and the bounds associated with the adopted random pricing strategy. These two together help to attain the equilibrium properties in Proposition 3.3.

Lemma 3.1. The following equation holds in equilibrium:

$$I_{1T} = I_{2T} \geq P^*_{2,T}, \quad (7)$$

Lemma 3.2. The following equation holds in equilibrium:

$$U_{1T} = U_{2T} = P^*_{2,T}. \quad (8)$$

Proposition 3.3. In the selling game, $\Gamma(2, q, T)$, each seller adopts the pricing strategy as $p = P^*_{2,T}$ for all $T \geq 2$, and the expected profit for each seller is exactly $\delta V(\bar{p}, 1, T)$.

The proof is presented in the appendix. As a result, when there are two firms competing on price to sell their product, they both offer an identical price equal to $P^*_{2,T}$. This equilibrium arises because neither party intends to sell below their option-value. Letting the other firm sell results in an equal expected profit for both due to equation (6).

3.4 Oligopoly Competition with Binary Demand

In this section, we examine the equilibrium when there are more than two firms in the market. First, we introduce some notations for this part:

- $F_{iNT}(p)$: the probability that seller i sets a price less than or equal to p in period T in a market containing N sellers,
- $F_{-iNT}(p)$: the probability at least one seller except the seller i set a price less than or equal to p in period T in a market containing N sellers,
- $F_{iNT}(p_-) = \lim_{p \to p^-} F_{iNT}(p)$,

- $\gamma_{NT}(p)$: the number of firms offering a price equal to p,
- $I_{iNT} = \sup\{p \mid F_{iNT}(p) = 0\}$,
- $U_{iNT} = \inf\{p \mid F_{iNT}(p) = 1\}$.

Now, we discuss the equilibrium properties of this oligopoly market in the following Proposition.
Proposition 3.4. When there are N firms in the market and the demand structure is binary, the three following conditions hold in equilibrium:

1. $I_{iNT} \geq P^*_{N,T}$.
2. $\gamma_{NT}(p^*_{N,T}) \geq 2$.
3. The expected profit for each firm is $V(\bar{P}, N, T)$.

According to the Proposition, in a binary demand structure, it's feasible to offer a price higher than the reservation price, leading to unpredictable price dispersion. However, as outlined in (Baye and Morgan, 2004) and discussed in (Sun, 2017), when only marginal prices are taken into account, this oligopoly model doesn't exhibit any price dispersion.

3.5 General Demand Model

In this section, we have a generalization of the cases that we discussed before. As a result, we investigate a general demand structure and investigate the duopoly and oligopoly models' equilibria. We start by introducing a few extra notations:

- q_i : The probability that the demand equals to i,
- $F(p)$: The identical assumed strategy chosen by all of the firms,
- $F_{iNT}(p)$: The probability that the seller i sets a price less than or equal to p in period T in a market containing N sellers,
- $F_{iNT}(p_-) = \lim_{p \to p^-} F_{\mathrm{iNT}}(p)$,
- $I_{iT} = \sup\{p \mid F_{iT}(p) = 0\}$,
- $U_{iT} = \inf\{p \mid F_{iT}(p) = 1\}$.

Now, similar to the binary demand section, we first calculate the option-value and reservation price. The option-value can be calculated from the following equation:

$$V(\bar{p}, N, T) = \sum_{i=0}^{N-1} q_i \delta V(\bar{p}, N-i, T-1) + \sum_{i=N}^{+\infty} q_i \bar{p}. \tag{9}$$

The first component of (9) is the expected profit we get if the demand is less than the number of players, and the second one represents the expected profit of the cases where demand is more than the number of players. In addition, since $P^*_{N,T}$ is the lowest price that sellers are willing to offer, it satisfies the following equations:

$$V(\bar{p}, N, T) = q_0 \delta V(\bar{p}, N, T-1) + (1 - q_0) P^*_{N,T}. \tag{10}$$

So, we have:

$$P^*_{N,T} = \frac{V(\bar{p}, N, T) - q_0 \delta V(\bar{p}, N, T-1)}{1 - q_0} = \frac{\sum_{i=1}^{N-1} q_i \delta V(\bar{p}, N - i_1, T-1) + \sum_{i=N}^{+\infty} q_t \bar{p}}{1 - q_0} \tag{11}$$

Equation (11) has an intuitive explanation similar to the one mentioned by (Sun, 2017) for the perishable assets. It means that if demand is not zero, $P^*_{N,T}$ is equal to the expected option-value. The monopolist's option-value can be calculated from the following equation:

$$V(\bar{p}, 1, T) = q_0 \delta V(\bar{p}, 1, T-1) + (1 - q_0) \bar{p}. \tag{12}$$

Using the equation (12), a closed-form of $V(\bar{p}, 1, T)$ can be derived as follows:

$$V(\bar{p}, 1, T) = (1 - q_0)\bar{p} + q_0 \delta (1 - q_0)\bar{p} + \cdots + q_0 \delta^{T-1} (1 - q_0)\bar{p} = (1 - q_0)\bar{p} \frac{1 - (q_0 \delta)^T}{1 - q_0 \delta}. \tag{13}$$

In the following sections, we first analyze the equilibrium for a duopoly and then for an oligopoly market.

3.6 Duopoly Competition with General Demand Function

To examine the properties of equilibrium, we need to first calculate the option-value. From the option-value we obtained for monopolist, the option-value for a firm in a duopoly market can be calculated from the following equations:

$$V(\bar{p},2,1) = (1 - q_0 - q_1)\bar{p}, \tag{14}$$

and we have:

$$V(\bar{p},2,T) = \sum_{l=0}^{1} q_i \delta V(\bar{p}, 2-i, T-1) + \sum_{i=2}^{+\infty} q_i \bar{p}, \tag{15}$$

for $T > 2$. The right-hand side of equation (15) can be calculated using the option-value of $T = 1$ and the monopolist option-value in (13). The following Proposition introduces the symmetric mixed strategy equilibrium for the duopoly case. A similar result exists and is proved in (Sun, 2017) for non-perishable assets. The proof of this Proposition is stated in the section appendix.

Proposition 3.5. Considering $q_1 > 0$ and $\sum_{l=2}^{+\infty} q_l > 0$, the mixed strategy $W_T(p)$ represents a Nash equilibrium, and its associated expected profit is $V(\bar{p}, N, T)$.

$$W_T(p) = \begin{cases} 0 & p \leq P^*_{N,T} \\ \dfrac{V(\bar{p},2,T) - q_0 \delta V(\bar{p},2,T-1) - \sum_{i=1}^{+\infty} q_i p}{q_1(\delta V(\bar{p},1,T-1) - p)} & P^*_{N,T} < p < \bar{p} \\ 1 & p \geq \bar{p} \end{cases} \tag{16}$$

In Proposition 3.5, we show that $W_T(p)$ can result in an equilibrium. Another characteristic we discovered for the duopoly case is the equilibrium's uniqueness. To show that this is a unique equilibrium, we take advantage of the expected profit in equilibrium and the equilibrium strategy. The evidence we need to demonstrate the uniqueness of $W_T(p)$ can be found in the following lemmas.

Lemma 3.6. In equilibrium the following equations hold:

$$I_{1T} = I_{2T} = P^*_{2,T}, U_{1T} = U_{2T} = \bar{P}. \tag{17}$$

Lemma 3.7. The equilibrium strategy is symmetric.

Building upon these foundational lemmas, Proposition 3.8 emerges as the centerpiece of our analysis. We demonstrate that W_T(p) stands as the unique mixed-strategy equilibrium, thus solidifying our understanding of equilibrium dynamics within the duopoly model.

Proposition 3.8. $W_T(p)$ is a unique mixed-strategy equilibrium.

3.7 Oligopoly Competition with General Demand Function

We define the function $Z_{K,N}(x)$ according to the following equation:

$$Z_{K,N}(x) = \sum_{i=0}^{K} \binom{N-1}{i} (1-x)^{N-l-1} x^l. \tag{18}$$

If all of the firms choose an identical strategy $F(p)$, the probability of an observation in which at most k firms offer a price less than or equal to p is $Z_{K,N}(F(p))$. Same as the duopoly case, we can again calculate the option-value as follows:

$$V(\bar{p},N,1) = \sum_{l=N}^{+\infty} q_l \bar{p} \tag{19}$$

Thus, having $V(\bar{p},N,1)$ and monopolist option-value, the values of $P^*_{N,T}$ and $V(\bar{p},N,1)$ can be calculated. Similar to oligopoly cases in (Sun, 2017) for non-perishable assets, the following statement proposes a symmetric mixed strategy equilibrium.

Proposition 3.9. With the conditions $\sum_{i=2}^{+\infty} q_i > 0$, when every firm chooses $W_{N,T}(p)$ as its strategy, this condition results in a unique symmetric equilibrium, and their expected profit in equilibrium is $V(\bar{P}, N, T)$:

$$G(x) = \frac{\sum_{l=1}^{+\infty} q_i \bar{p} + \sum_{l=1}^{n-1} q_i z_{i-1}(x) \delta V(\bar{P}, N-i, T-1)}{\sum_{l=1}^{+\infty} q_i + \sum_{i=1}^{n-1} q_i Z_{i-1}(x)}, W_{N,T}(p) = \begin{cases} 0 & p \le P_{N,T}^* \\ G^{-1}(p) & P_{N,T}^* < p < \bar{p} \\ 1 & p \ge \bar{p} \end{cases} \tag{20}$$

As it was mentioned before, it is important to examine the uniqueness of equilibrium. Therefore, the following lemma is employed in proposition 3.11 to demonstrate the uniqueness of equilibrium under some particular circumstances. For this aim, we only need to provide a situation in which asymmetric equilibrium is not possible because we have shown that the only possible symmetric strategy is $W_{N,T}(p)$.

Lemma 3.10. With the conditions $\sum_{i=2}^{+\infty} q_l > 0, W_T(p)$ is a right continuous increasing function at $p \in (P_{N,T}^*, \bar{p})$.

Proposition 3.11. If the conditions $\sum_{l=2}^{+\infty} q_i > 0$ and (23) are satisfied, the equilibrium should be symmetric.

$$p_{N,T}^* \le I_{1NT} = l_{2NT} = \cdots = l_{NNT} \tag{21}$$

Proposition 3.11 provides valuable insights into the equilibrium dynamics of our model. By satisfying the specified conditions and ensuring symmetry in equilibrium, we gain a deeper understanding of the system's stability and behavior.

4. Numerical Results

In this section, we delve into the analysis of results obtained in previous chapters and explore the influence of various parameters. Utilizing the derived distribution functions, we investigate equilibrium strategies. Additionally, we present numerical examples and graphs to aid comprehension. Initially, as part of the model evaluation and validation process, we examine the performance of the obtained models over infinite planning horizons, comparing outcomes with those of Sun (2017).

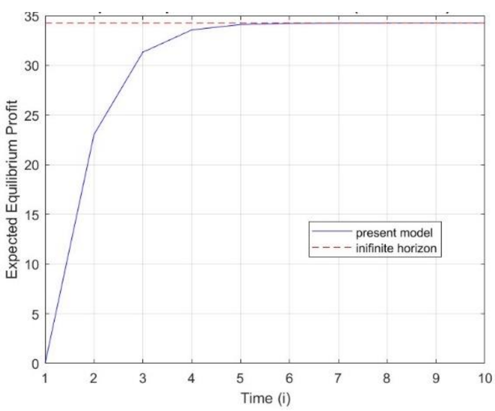

Figure 1. Duopoly model with Binary Demand

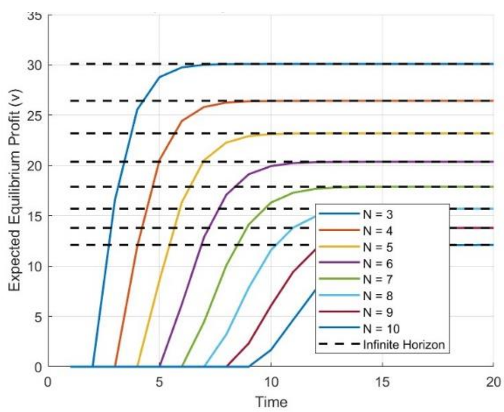

Figure 2. Oligopoly model with Binary Demand

In these experiments, we consider $\delta = 0.9$ and $\bar{P} = 40$. Here, in figure 1, we have obtained the equilibrium price of duopoly competition from our model for $T = 1, \dots, 10$ and the associated result from the model in (Sun, 2017). Same experiment is conducted for oligopoly model with $N = 3, \dots, 10$, and $T = 1, \dots, 10$. The effective prices in equilibrium are compared in Figure 2. In both these figures, we observe that the result of presented model converges to the output of the infinite horizon model as T increases. For general demand, as we showed in the preceding sections, when $\sum_{l=2}^{+\infty} q_l > 0$, then we have a mixed-strategy equilibrium. In Figures 3 and 4 , we have the equilibrium distribution of our model for $N = 2$ and $N = 3$, respectively. We considered that the demand follows a poison distribution with expected value of 0:5. Similarly, we observe that the distribution of the price converges to the equilibrium distribution associated to infinite horizon as T increases.

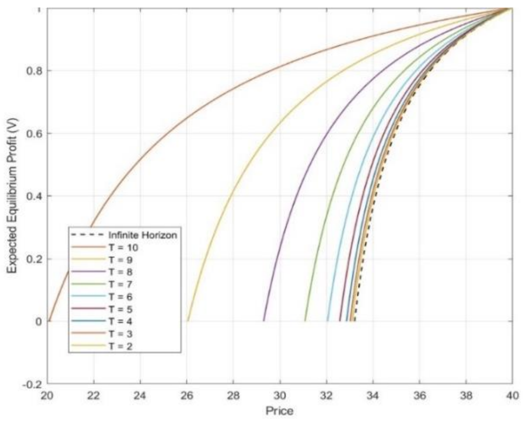

Figure 3. Duopoly model with General Demand

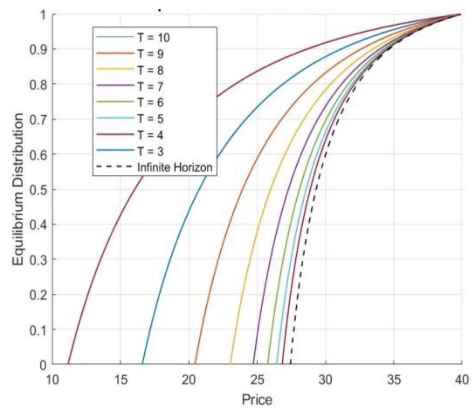

Figure 4. Oligopoly model with General Demand

5. Conclusion

This study outlines four pricing models with limited capacity and identifies equilibrium strategies. The strategies are generally symmetric except in the case of duopoly with binary demand. It demonstrates that under efficient price criteria, there's zero price dispersion when demand is in balance, but in case of excess demand, the market lacks a single price and follows a predictable pattern instead. These equilibrium strategies can empower sellers of limited capacity perishable goods to maximize profits. They tend to set lower prices to attract demand when it's uncertain and avoid the risk associated with higher prices. As the planning horizon extends, the results converge towards a previously established model, suggesting its reliability for longer-term planning. Each vendor in these models can only offer one product, but future research could explore broader capacities for a more realistic depiction of market dynamics. The study doesn't conclusively establish a unique equilibrium strategy in oligopoly models with general demand, leaving room for future investigations into rejecting asymmetric strategies. Additionally, it suggests that sellers conceptualize the option-value before it's quantified mathematically, likening it to the potential profits of monopolization. This conceptualization may vary across markets, prompting adjustments to the analysis depending on specific circumstances.

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The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.

Data sharing statement

No additional data are available.

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Appendix A

To have clarity in the proof process, we first introduce some new notations here.

1. Duopoly competition with binary demand: The equilibrium expected profit of firm i is denoted by $\bar{V}_i^2(q,T)$. In addition the expected profit of firm i when firm j 's strategy F_{jT} is given, is indicated by $V_i^2(p;q,T)$ and is calculated as follows:

$$V_i^2(p;q,T) = q\delta\bar{V}_i^2(q,T-1) + (1-q)\big[F_{jT}(p_-)V(\bar{P},1,T-1) + \left(F_{jT}(p) - F_{jT}(p_-)\right)\frac{1}{2}(\delta V(\bar{P},1,T-1)+p) + \left(1-F_{jT}(p)\right)p\big]. \tag{22}$$

2. Oligopoly competition with binary demand: The equilibrium expected profit of firm i is denoted by $\bar{V}_i^N(q,T)$. The expected profit of firm i when F_{-iNT} is given, is indicated by $V_i^N(p;q,T)$ and is calculated as follows:

$$V_i^N(p;q,T) = q\delta\bar{V}_i^N(q,T-1) + (1-q)[F_{-iNT}(p_-)\delta V(\bar{P},N-1,T-1) + \left(F_{-iNT}(p) - F_{-iNT}(p_-)\right)\left(\frac{m}{m+1}\delta\bar{V}_i^{N-1}(q,T-1) + \frac{1}{m+1}p\right) + (1-F_{-iNT}(p))p]. \tag{23}$$

The parameter m can be calculated using the joint distribution of all $N-1$ firms. For our purpose, it is enough to know that it is a scalar in $[1, N-1]$.

3. Duopoly competition with general demand: The expected value of the profit of the firm i when firm j 's strategy F_{jT} is given, is indicated by $W_i^2(p;q,T)$ and is calculated as follows:

$$W_i^2(p;q,T) = q_0\delta\bar{W}_i^2(q,T-1) + q_1\big[F_{jT}(p_-)\delta V(\bar{P},1,T-1) + \left(F_{jT}(p) - F_{jT}(p_-)\right)\frac{1}{2}(V(\bar{P},1,T-1)+p) + \left(1-F_{jT}(p)\right)p\big] + \sum_{i=2}^{+\infty} q_i p, \tag{24}$$

Here, we introduce a crucial theory that is applied throughout the proofs from (Baye & Morgan, 2004).

Theorem 4.1. In a scenario where a game involves an infinite set of strategies, the combination of strategies $\{F_1, \dots, F_N\}$ forms a mixed strategy Nash equilibrium if and only if, for every player i, no action within their strategy set results in a payoff higher than their equilibrium payoff when considering the strategies of all other players $\{F_1, \dots, F_N\} \setminus F_i$. Additionally, the likelihood of actions yielding a payoff lower than the equilibrium payoff for player i, given the strategies of other players $\{F_1, \dots, F_N\} \setminus F_i$, must be zero.

Let's now explore the proofs presented for the equilibrium properties. Given that several equilibrium traits are analogous in both finite and infinite scenarios, we may frequently adopt the methodologies outlined in (Sun, 2017) for the infinite case to demonstrate similar characteristics within our context. However, it's essential to note that while following similar steps, the specifics of our setting may lead to different outcomes.

A. 1 Proof of Lemma 3.1.

Proof. Let us use proof by contradiction. First, we assume $I_{1T} < I_{2T}$. Considering this assumption, firm 1 can increase its profit by assigning the probability of the interval (I_{1T}, I_{2T}) to a point in the left neighborhood of I_{2T}. The reason is that they are sure that all of these prices would be sold with probability one if there exists any demands, thus they try to choose the maximum of them. So, this is a contradiction arising from our false assumption, and we have $I_{1T} = I_{2T}$ in equilibrium. In addition, if we assume $I_{1T} = I_{2T} < P_{2,T}$ in equilibrium, there will be two possible states, and we demonstrate that neither of them can lead to equilibrium. Therefore, we prove that the equation $I_{1T} = I_{2T} < P_{2,T}$ cannot stand in equilibrium.

Assume $F_{2T}(I_{2T}) < 1$. Because the cumulative distribution function is right-continuous, we have:

$$\exists\eta \in \left(l_{2T}, P_{2,T}^*\right) \mid \forall p \in [I_{2T}, \eta) \to F_{2T}(I_{2T}) \le F_{2T}(p) < 1. \tag{25}$$

Now, considering firm 2's strategy, firm 1 achieves smaller profit for $p \in [I_{2T}, \eta)$ than deciding not to sell in this period. Utilizing equation (6) and $p < P_{2,T}^* = \delta V(P,1,T-1)$, we have:

$$V_1^2(p;q,T) < q\delta\bar{V}_1^2(q,T-1) + (1-q)P_{2,T}^*. \tag{26}$$

The inequality (26) is obtained by substituting p by its upper bound $\delta V(\bar{P},1,T-1)$ in (22), and the right-hand side demonstrates the expected profit of firm 1, when they allow the other seller to sell. As a result, based on Theorem 4.1, it is not possible to have a pure strategy with a better result than the equilibrium's one. So, firm 1 never offers $p \in [I_{2T},\eta)$, and this is a contradiction. So, state 1 is not possible in equilibrium. Assume $F_{2T}(I_{2T}) = 1$ in equilibrium. Then, firm 2's expected profit associated with $p = I_{2T}$ is absolutely less than its corresponding value for postponing the selling. This is shown in the following inequality:

$$V_2^2(p;q,T) < q\delta\bar{V}_2^2(q,T-1) + (1-q)P_{2,T}^*. \tag{27}$$

The right-hand side of (27) is obtained based on the assumption $I_{2T} < P_{2,T}^*$ and by substituting p in (22) with its upper bound $P_{2,T}^* = \delta V(\bar{P},1,T-1)$. Because firm 2 has the option to increase its profit by choosing not to sell, this state cannot be an equilibrium according to the definition of a Nash equilibrium.

A. 2 Proof of Lemma 3.2.

Proof. First, we assume $U_{1T} \neq U_{2T}$ and $U_{1T} < U_{2T}$. Then, from Lemma 3.1, we have $P_{2,T}^* \leq U_{1T} < U_{2T}$. If $U_{1T} < U_{2T}$ in equilibrium, there will be four possible states, and we demonstrate that none of them can lead to equilibrium. In other words, we prove that the equation $U_{1T} \neq U_{2T}$ is not possible in equilibrium.

1. $P_{2,T}^* < U_{1T} < U_{2T}$: Considering $p_1 \in \left(P_{2,T}^*, U_{1T}\right)$ and $p_2 \in (U_{1T}, U_{2T})$ the following equation holds:

$$V_2^2(p_1;q,T) > q\delta\bar{V}_2^2(q,T-1) + (1-q)\delta V(\bar{P},1,T-1) = V_2^2(p_2;q,T). \tag{28}$$

The inequality is obtained from the substitution of p_1 with its lower bound $P_{2,T}^* = \delta V(\bar{P},1,T-1)$ in the function $V_2^2(p_1;q,T)$. Therefore, firm 2's expected profit for (U_{1T}, U_{2T}) is less than the equilibrium profit, and based on Theorem 4.1 this interval should be assigned zero probability. This is in contradiction with our assumption $P_{2,T}^* < U_{1T} < U_{2T}$.

2. $P_{2,T}^* = U_{1T} < U_{2T}$: From lemma 3.1, we can conclude $P_{2,T}^* = I_{1T} = U_{1T} = I_{2T}$. For $p_1 \in \left(P_{2,T}^*, U_{2T}\right)$, the following inequality holds:

$$V_1^2(p_1;q,T) > q\delta\bar{V}_1^2(q,T-1) + (1-q)\delta V(\bar{P},1,T-1) = V_1^2\left(P_{2,T}^*;q,T\right). \tag{29}$$

We have (29) by substituting p_1 with its lower bound $P_{2,T}^* = \delta V(\bar{P},1,T-1)$ in (22). Therefore, firm 1's profit for $p_1 \in \left(P_{2,T}^*, U_{2T}\right)$ is greater than its profit for $P_{2,T}^*$. Thus, it is in contradiction with the definition of Nash equilibrium, and this state cannot be an equilibrium.

As a result of rejection of state 1 and 2 , we can conclude that the equation $U_{1T} = U_{2T}$ should stand in equilibrium.

3. Assume $U_{1T} = U_{2T} > I_{1T} = I_{2T} = P_{2,T}^*$. Because cumulative distribution function is right-continues, we have:

$$\exists\eta \in \left(P_{2,T}^*, U_{2T}\right) \mid \forall p \in [I_{2T},\eta] \rightarrow F_{2T}(p) < 1. \tag{30}$$

So, for $p = \eta$, we have:

$$V_1^2(p;q,T) > q\delta\bar{V}_1^2(q,T-1) + (1-q) + (1-q)\delta V(\bar{P},1,T-1). \tag{31}$$

The inequality results from $p > P_{2,T}^* = \delta V(\bar{P},1,T-1)$, and the right-hand side is equal to the expected profit associated with $P_{2,T}^*$. In addition, for the points in a small right neighbourhood of $P_{2,T}^*$, we have:

$$V_1^2(p;q,T) > q\delta\bar{V}_1^2(q,T-1) + (1-q) + (1-q)\delta V(\bar{P},1,T-1), \tag{32}$$

which is obtained from the fact that $p \rightarrow P_{2,T}^* = \delta V(\bar{P},1,T-1)$ and its substitution in $V_1^2(p;q,T)$. Based on limit definition, there exists $\epsilon > 0$ as the expected profit of $P \in \left[p_{2,T}^*, P_{2,T}^* + \epsilon\right]$ is less than the expected profit of $p = \eta$. This means that there is a pure strategy with the expected profit greater than the equilibrium one, and based on Theorem 4.1, this state cannot be an equilibrium. In other words based on Nash equilibrium definition, firm 1 never offers a price in this interval, and we have:

$$I_{1T} = \sup\{p \mid F_{1T}(p) = 0\} > P^*_{2,T}, \tag{33}$$

and this is in contradiction with our initial assumption regarding the lower bound I_{1T}.

4. Assume $U_{1T} = U_{2T} \geq I_{1T} = I_{2T} > P^*_{2,T}$. If the probability firm 2 assigned to U_{2T} is not positive, for each $\epsilon > 0$, there exists $\eta \in \left(P^*_{2,T}, U_{2T}\right)$ as:

$$\forall p \in (\eta, U_{2T}] \rightarrow F_{2T}(p) > 1 - \epsilon. \tag{34}$$

When we consider $\epsilon = \frac{I_{1T} - P^*_{2,T}}{\frac{1}{2}(\delta V(\bar{P},1,T-1) + 3U_{2T})}$, the expected profit of firm 1 for p $\in (\eta, U_{2T}]$ satisfies the following inequality:

$$\begin{aligned} V_1^2(p;q,T) &< q\delta\bar{V}_1^2(q,T-1) + (1-q)\left[\delta V(\bar{P},1,T-1) + \frac{1}{2}\epsilon(\delta V(\bar{P},1,T-1)) + \frac{3}{2}U_{2T}\right)\right] \\ &= q\delta\bar{V}_1^2(q,T-1) + (1-q)I_{1T}. \end{aligned} \tag{35}$$

The inequality is obtained by substituting $1 - F_{2T}(p), F_{2T}(p) - F_{2T}(p_-)$and p with their upper bounds, $1, \epsilon$ and U_{2T} in $V_1^2(p;q,T)$, the following inequality is derived: The last expression is obtained by substitution of ϵ. On the other hand, in a left neighborhood of I_{1T}, the expected profit is:

$$\lim_{p \to I_{1T}^-} V_1^2(p;q,T) = q\delta\bar{V}_1^2(q,T-1) + (1-q)I_{1T} \tag{36}$$

So, in a left neighborhood of I_{1T}, the expected profit is more than the expected profit of $p \in (\eta, U_{2T}]$. Thus, based on theorem 4.1, this state is not possible in equilibrium. So, we should have $F_{2T}(U_{2T}) - F_{2T}(U_{2T-}) > 0$. In this case, since we have $U_{1T} > P^*_{2,T} = \delta V(\bar{P},1,T-1)$, the value of $V_1^2(p;q,T)$ in a small left neighborhood of U_{1T} is greater than its value in U_{1T}. Accordingly, based on Theorem 4.1, the probability given to the points with expected value below the equilibrium expected value is zero, and based on this, the probability firm 1 assigns to U_{1T} is zero. However, we proved that the probability assigned to the upper bound should not be zero. As a result, we showed that state 4 is not possible to maintain equilibrium.

A. 3 Proof of Proposition 3.3

Proof. Considering Lemma 3.1 and 3.2, for a duopoly competition with binary demand and $T \geq 2$, the necessary condition for equilibrium is to have a pure strategy $I_{1T} = I_{2T} = U_{1T} = U_{2T}$. Since we have $P^*_{2,T} = \delta V(\bar{P},1,T-1)$, it can be shown that this strategy is a Nash equilibrium. When $T = 1$, this is a simple Bertrand model in which every firms offers zero in equilibrium.

A. 4 Proof of Proposition 3.4

Proof. The proof of this Proposition is similar to the proof of Proposition 3.3. These conditions are proved to be true for $N = 2$. So, we can use inductive reasoning. Assume that the three sections of the Proposition are valid for $\Gamma(N-1,q,T)$. Let us first demonstrate $I_{iNT} \geq P^*_{N,T}$. Without loss of generality, let us first consider the following equation:

$$I_{1NT} = \min_{1 \leq i \leq N} I_{iNT} \tag{37}$$

Now, we use proof by contradiction. For this aim, we assume $I_{1NT} < P^*_{N,T}$, and we examine the two resulting possibilities.

1. If $F_{-1NT}(I_{1NT}) < 1$, from right-continuity property of cumulative distribution function, we have:

$$\exists \eta \in \left(I_{1NT}, P^*_{N,T}\right) \mid \forall p \in [I_{1NT}, \eta] \rightarrow F_{-1NT}(p) < 1. \tag{38}$$

Now, consider $p \in [I_{1NT}, \eta]$. Since we have $p < \delta V(\bar{P}, N-1, T-1) = P^*_{N,T}$, if we substitute p in (26) with the upper bound $\delta V(\bar{P}, N-1, T-1)$, we have the following inequality:

$$V_1^N(p;q,T) < q\delta\bar{V}_i^N(q,T-1) + (1-q)\delta V(\bar{P},N-1,T-1). \quad (39)$$

The right-hand side of (39) is firm 1's expected profit if they decide to postpone their selling. So, based on Theorem 4.1, the probability assigned to the points with the expected profit less than the equilibrium expected profit must be zero in equilibrium, and as a result it is not possible that firm 1 post a price in $[I_{1NT},\eta]$ in equilibrium. This is a contradiction, and this state is rejected.

2. Assume $F_{-1NT}(I_{1NT}) = 1$. Because we assumed that I_{1NT} is the least lower bound, there is at least one firm, other than 1 , that posts a price equal I_{1NT} with probability one. As a result, there exists firm $i \neq 1$, such that $F_{iTN}(I_{1NT}) - F_{iTN}(I_{1NT}^{-}) > 0$. Since we have $I_{1TN} < p_{N,T}^{*} = \delta V(\bar{P},N-1,T-1)$, and $I_{iNT} = I_{1NT}$ is the least lower bound, if we substitute I_{1NT} with its upper bound $\delta V(\bar{P},N-1,T-1)$ in (23), we have:

$$V_i^N(p;q,T) < q\delta\bar{V}_i^N(q,T-1) + (1-q)\delta V(\bar{P},N-1,T-1). \quad (40)$$

The right-hand side is the expected value of firm i 's profit when they postpone selling. So, firm i should not assign positive probability to I_{iNT}, and this is a contradiction.

Now, the first part of Proposition is proved. We again use proof by contradiction to prove the second part $\gamma_{NT}(p_{N,T}^{*}) \geq 2$. We first assume:

$$p_{N,T}^{*} \leq I_{1NT} < I_{2NT} \leq \cdots \leq I_{NNT}. \quad (41)$$

Considering this assumption, we have:

$$\exists \eta \in (I_{1NT}, I_{2NT}) \mid F_{1NT}(\eta) > 0. \quad (42)$$

We only need to calculate firm 1's expected profit for $p \in [I_{1NT},\eta]$. Since firm 1 offers the lowest price in the market, it can sell for sure if demand is not zero. So, we have:

$$V_i^N(p;q,T) = q\delta\bar{V}_i^N(q,T-1) + (1-q)p. \quad (43)$$

The right-hand side of (43) is bounded from above with its value for $p = \eta$. As a result, offering a price equal to $p \in [I_{1TN},\eta)$ is not possible in equilibrium, and it is in contradiction with our assumption. As a result, the following relation stands:

$$P_{N,T}^{*} \leq I_{1NT} = I_{2NT} \leq \cdots \leq I_{NNT}. \quad (44)$$

To prove the second part, we should also show $P_{N,T}^{*} < I_{1NT} = I_{2NT}$ is not possible in equilibrium. We use proof by contradiction. Let us assume that $P_{N,T}^{*} < I_{1NT} = I_{2NT}$ in equilibrium. Now, we examine and reject the two possible resulting states.

1. Assume $F_{-1NT}(I_{1NT}) > 0$. Then, since I_{1NT} is greater than $\delta V(\bar{P},N-1,T-1)$, firm 1 's expected profit satisfies the following equation:

$$V_1^N(I_{1NT};q,T) < q\delta\bar{V}_1^N(q,T-1) + (1-q)I_{1NT}. \quad (45)$$

We also have the following inequality and equality for the limit of $V_1^N(p;q,T)$ when p approaches to I_{1NT} from its right and left neighbourhoods, respectively:

$$\lim_{p\to J_{1NT}+} V_1^N(p;q,T) < q\delta\bar{V}_1^N(q,T-1) + (1-q)I_{1NT}, \quad (46)$$

$$\lim_{p\to I_{1NT}-} V_1^N(p;q,T) = q\delta\bar{V}_1^N(q,T-1) + (1-q)I_{1NT}.$$

Based on the limit definition, we can find ϵ_1 and ϵ_2 as the expected profit of $p \in [I_{1NT}, I_{1NT}+\epsilon_1)$ is less than the expected profit of $p \in (I_{1NT}-\varepsilon_2, I_{1NT})$. So, based on Theorem 4.1, firm 1 never offers a price from $[I_{1NT}, I_{1NT}+\epsilon_1)$

in equilibrium, and we can reject the assumption $F_{-1}(I_{1NT}) > 0$.
2. Assume $F_{-1NT}(I_{1NT}) = 0$. In this situation, every firm has the option to sell at I_{1NT} when there is a positive demand. So, since $I_{1NT} > V_1^N(p;q,T) = P^*_{N,T}$, the expected profit of firm $i, V_1^N(p;q,T)$, for $p = I_{1NT}$ is strictly greater than its corresponding value for selling postponement. Thus, there is no firm tend to let others sell, and the following expression holds:

$$U_{1NT} = U_{2NT} = \cdots = U_{NNT}. \tag{47}$$

In addition, the probability assigned to the upper bound is zero for all firms. In order to show this we examine two different possibilities. First, if firm i assigns a positive probability to U_{iNT}, and there exists at least one another firm j who assigns zero probability to U_{jNT}, firm i 's expected profit for U_{iNT} is equal to the expected profit of postponement. So, from Theorem 4.1 the probability firm i assigns to U_{iNT} should be zero. This is a contradiction.
Secondly, If all of the firms assign positive probability to the upper bound, since $\delta V(\bar{P}, N-1, T-1) < U_{iNT}$, the expected profit of firm i at U_{iNT} is less than its corresponding value for a small left neighborhood of U_{iNT}. This is because in the left neighborhood, it has more chance to sell. So, it has to assign zero probability to

U_{iNT}, and this is a contradiction. So, we showed that the upper bounds are equal, and the probability assigned to them is zero. As a result, the expected value of profit for firm 1 satisfies the following equation:

$$\lim_{p \to U_{1NT}^-} V_1^N(p;q,T) = q\delta\bar{V}_1^N(q,T-1) + (1-q)\delta V(\bar{P}, N-1, T-1). \tag{48}$$

Since we have $\delta V(\bar{P}, N-1, T-1) < I_{1NT}$, we also have:

$$\lim_{p \to U_{1NT}^-} V_1^N(p;q,T) < q\delta\bar{V}_1^N(q,T-1) + (1-q)I_{1NT}. \tag{49}$$

The right-hand side is the expected profit of firm 1 for $p = I_{1NT}$. So, the probability assigned to a left neighborhood of U_{1NT} must be zero, and this is a contradiction with the upper bound definition. So, we have the following result is established:

$$I_{1NT} = I_{2NT} = P^*_{N,T} \tag{50}$$

Applying proof by contradiction, we show that at least two firms assign probability one to $P^*_{N,T}$. Assume that non of the firms assign probability one to $P^*_{N,T}$. First, we have that the limit of the expected profit of firm 1 when p approaches to $I_{1NT} = P^*_{N,T}$ from its right neighborhood is equal to its corresponding value at $P^*_{N,T}$. In other words, we have:

$$\lim_{p \to I_{1NT}^+} V_1^N(p;q,T) = q\delta\bar{V}_1^N(q,T-1) + (1-q)P^*_{N,T}. \tag{51}$$

Secondly, we can find η such that:

$$\exists \eta > p^*_{N,T} \mid 0 < F_{-1NT}(\eta) < 1. \tag{52}$$

Now, because $P^*_{N,T} = \delta V(\bar{P}, N-1, T-1) < \eta$, firm 1's expected profit satisfies the following inequality:

$$V_1^N(\eta;q,T) > q\delta\bar{V}_1^N(q,T-1) + (1-q)P^*_{N,T} \tag{53}$$

As a result of (52) and (53) and based on the limit definition, parameter ϵ and interval $[I_{1NT}, I_{1NT} + \epsilon)$ can be found such that for each price in this interval the expected profit of firm 1 is less than the expected profit of η. So, this state is not possible in equilibrium. Therefore, we should have $F_{-i}\left(P^*_{N,T}\right) = 1$. Since we assume that the mixed strategy randomizations are independent in definition of mixed strategy (Baye & Morgan, 2004), There should be a firm j that offers I_{1NT} with probability one. With the same reasoning for firm j, there should be another firm i that posts I_{1NT} with probability one, and the result is established. Therefore, the second part of equation is proved.
We can infer from the preceding sections that the expected profit is equal to $V(\bar{P}, N, T)$. Let us first prove it for $T = N$. We have $P^*_{N,N-1} = V(\bar{P}, N, N-1) = 0$, and based on the previous sections two firms offer zero price. So, we have:

$$V_i^N\left(P^*_{N,T};q,N\right) = q\delta 0 + (1-q)\delta V(\bar{P}, N-1, T-1) = V(\bar{P}, N, T). \tag{54}$$

So, the induction reasoning can be applied. Note that there are two induction assumptions we are making to prove the third part of the proposition. First, we start with the assumption we made regarding a game with $N-1$ firms, and that is applied in all three sections' proofs. In the meantime, we made another assumption regarding a game with the planning horizon $T-1$ to prove the third section of the Proposition. As a result, we now assume that the third statement is valid for $T-1$, and we calculate the expected profit for $\Gamma(N,T,q)$ as follows:

$$V_i^N\left(P_{N,T}^*;q,T\right)=q\delta V(\bar{P},N,T-1)+(1-q)\delta V(\bar{P},N-1,T-1) \tag{55}$$

Thus, the result is established.

A. 5 Proof of Proposition 3.5

Proof. Let us prove this by induction. First, we must establish that $W_1(p)$ leads to equilibrium, and the equilibrium profit equals $V(\bar{p},2,1)$. Substituting $V(\bar{p},1,0)=0$ in the given formula for $W_1(p)$, we have:

$$W_1(p)=\begin{cases}0 & p\le P_{N,T}^*\\ \dfrac{\sum_{i=1}^{+\infty}\ q_ip-V(\bar{p},2,1)}{q_1p} & P_{N,T}^*<p<\bar{p}\\ 1 & p\ge\bar{p}\end{cases} \tag{56}$$

Then, if we substitute $F_{2T}(p)=w_1(p)$ in (28), we have:

$$W_1^2(p;q,1)=q_1p\left[\frac{V(\bar{p},2,1)+q_1(p)-\sum_{i=1}^{+\infty}\ q_ip}{q_1(p)}\right]+\sum_{i=2}^{+\infty}\ q_ip=V(\bar{p},2,1) \tag{57}$$

for $p\in(P_{2,1}^*,\bar{p})$, and $W_1^2(p;q,1)=0$ for other values of p. The expected value of profit for $p\in\left(P_{2,1}^*,\bar{p})\right)$ is similar.

In conclusion, no pure strategy has an expected profit greater than the equilibrium expected profit, and there is no probability ascribed to any pure strategy that has an expected profit that is less than the equilibrium profit. As a result, the necessary and sufficient conditions of equilibrium stated in Theorem 4.1 are satisfied. To use inductive reasoning, we assume that the conditions are satisfied for $T-1$. substituting $W_T(p)$ instead of $F_{2T}(p)$ in $W_1^2(p;q,T)$, we have:

$$\begin{aligned}W_1^2(p;q,T)&=q_0\delta V(\bar{p},2,T-1)+q_1\left[\frac{V(\bar{p},2,T)-q_0\delta V(\bar{p},2,T-1)-\sum_{i=1}^{+\infty}\ q_ip}{q_1(\delta V(\bar{p},1,T-1)-p)}\delta V(\bar{p},1,T-1)\right.\\&\left.+\left(\frac{q_1(\delta V(\bar{p},1,T-1)-p)-V(\bar{p},2,T)+q_0\delta V(\bar{p},2,T-1)+\sum_{i=1}^{+\infty}\ q_ip}{q_1(\delta V(\bar{p},1,T-1)-p)}\right)p\right]+\sum_{i=2}^{+\infty}\ q_ip=V(\bar{p},2,T)\end{aligned} \tag{58}$$

for $p\in\left(P_{2,T}^*,\bar{p}\right)$. In addition, based on the formulation of $W_T(p)$, we have $W_1^2(p;q,T)=0$ for $p\notin\left(P_{2,T}^*,\bar{p}\right)$. As a result, the necessary and sufficient of equilibrium in Theorem 4.1 are satisfied, and the result is established.

A. 6 Proof of lemma 3.6.

Let's start by demonstrating that $I_{1T}=I_{2T}$. To employ proof by contradiction, assume $I_{1T}<I_{2T}$. According to this assumption, there exists $\eta>I_{1T}$ such that $F_{1T}(\eta_-)>0$. In this case, since firm 2 is absent, firm 1 acts like a monopolist. Therefore, the expected profit of $p\in[I_{1T},\eta)$ is smaller than the expected value resulting from offering η. This interval should not have a positive probability in equilibrium according to the Theorem 4.1, which is a contradiction. So, we have $I_{1T}=\ I_{2T}$ in equilibrium. Now, we need to show $I_{1T}=I_{2T}=P_{2,T}$. We again use proof by contradiction. If we assume $I_{1T}=I_{2T}<P_{2,T}^*$, there will be two possible states. Here, we examine both of them and demonstrate why it is impossible for them to establish an equilibrium.

1. $F_{2T}(I_{2T})<1$: Because cumulative distribution function is right-continues, we have:

$$\exists\eta\in\left(I_{2T},P_{2,T}^*\right)\mid\forall p\in[I_{2T},\eta)\rightarrow F_{2T}(I_{2T})\le F_{2T}(p)<1. \tag{59}$$

Since when $\sum_{1=2}^{+\infty}q_i>0$, we have $P_{2T}^*>\delta V(\bar{P},1,T-1)$, we can substitute P_{2T}^* as an upper bound for both p and

$\delta V(\bar{P},1,T-1)$ in $W_1^2(p;q,T)$. As a result of this substitution, we obtain the following inequality:

$$W_1^2(p;q,T) < q_0\delta\bar{W}_1^2(q,T-1) + q_1P_{2,T}^* + \sum_{i=2}^{+\infty} q_iP_{2,T}^*. \tag{60}$$

for $p \in [I_{2T},\eta)$. From (9) and (10), we have:

$$(1-q_0)P_{N,T}^* = q_1\delta V(\bar{p},1,T-1) + \sum_{i=2}^{+\infty} q_i\bar{p}. \quad . \tag{61}$$

Now, we substitute $(1-q_0)P_{N,T}^*$ in (60) :

$$W_1^2(p;q,T) < q_0\delta\bar{W}_1^2(q,T-1) + q_1\delta V(\bar{P},1,T-1) + \sum_{i=2}^{+\infty} q_i\bar{P}. \tag{62}$$

The right-hand side of (62) demonstrates the expected value associated with offering the maximum price $\bar{P}$. So, the expected benefit of $p \in [I_{2T},\eta)$ is less than its value for $p = \bar{P}$. So, we found a pure strategy with expected profit greater than the equilibrium expected profit, and this state cannot establish an equilibrium based on Theorem 4.1.

2. Assume $F_{2T}(I_{2T}) = 1$. The intuition is comparable to the prior section. Because $P_{2,T}^* > \delta V(\bar{P},1,T-1)$ and by (9), firm 2 achieves less profit for $p = I_{2T}$ than $p = \bar{P}$:

$$W_2^2(p;q,T) < q_0\delta\bar{W}_2^2(q,T-1) + q_1P_{2T}^* + \sum_{i=2}^{+\infty} q_iP_{2T}^*. \tag{63}$$

So, refusing these two states, we have $I_{1T} = I_{ZT} \geq P_{2T}^*$ in equilibrium.

Now, we only need to show $p_{N,T}^* < I_{1T} = I_{ZT}$ is not possible to establish an equilibrium. We again employ proof by contradiction. Below, we examine the two possible states arising from our assumption $p_{N,T}^* < I_{1T} = I_{2T}$.

1. Assume $F_{2T}(I_{2T}) > 0$. Then, because we have $I_{1T} > P_{N,T}^*$ and $P_{N,T}^* > \delta V(\bar{P},1,T-1)$, substituting I_{1T} as an upper bound instead of $\delta V(\bar{P},1,T-1)$ in $W_1^2(I_{1T};q,T)$, we have:

$$W_1^2(I_{1T};q,T) < q_0\delta\bar{W}_2^2(q,T-1) + q_1I_{1T} + \sum_{i=2}^{+\infty} q_iI_{1T}. \tag{64}$$

By same intuition, when p approaches to I_{1T} from its right neighbourhood, we have the following inequality:

$$\lim_{P\to I_{1T^+}} W_1^2(q;T) < q_0\delta\bar{W}_2^2(q,T-1) + q_1I_{1T} + \sum_{i=2}^{+\infty} q_iI_{1T}. \tag{65}$$

On the other hand, when p approaches to I_{1T} from its left neighbourhood, the limit of expected profit is calculated as follows:

$$\lim_{P\to I_{1T^-}} W_1^2(q;T) = q_0\delta V_1(T-1) + q_1I_{1T} + \sum_{i=2}^{+\infty} q_iI_{1T}. \tag{66}$$

Based on limit definition, there are $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that for $p \in [I_{1T}, I_{1T} + \epsilon_1)$ the expected profit is less than the expected profit for $p \in (I_{1T} - \varepsilon_2, I_{1T})$. According to Theorem 4.1, firm 1 never assign a positive probability to this interval in equilibrium, and this is a contradiction.

2. Assume $F_{2T}(I_{2T}) = 0$. as a result of the prior state being rejected, we also have $F_{1T}(I_{1T}) = 0$. In this situation, both firms have an option to sell for I_{1T}. Based on our assumption $p_{N,T}^* < I_{1T} = I_{2T}$, and the relationship

we have in (61), the expected profit of postponing the sell is less than the existing option of offering $I_{1T} = I_{2T}$. Therefore, probability assigned to postponement is zero, and the following equation holds:

$$U_{1T} = U_{2T} \leq \bar{P}. \quad (67)$$

In addition, it can be shown that the probability assigned to upper bounds cannot be positive. This is due to the fact that if only one of them gives the upper bound a positive probability, that would be the same as giving the postponement a positive probability, which we have shown is impossible in equilibrium. Furthermore, it can be demonstrated that the expected profit of the upper bounds is lower than the expected profit associated with the prices in a left neighborhood of the upper bound, making it impossible for both of them to simultaneously assign a positive probability to the upper bound. The reason is that $V(\bar{P},1,T-1) < p_{N,T}^{*} < I_{1T} = I_{2T} \leq U_{1T} = U_{2T}$.

Therefore, when p approaches to U_{1T} from its left neighborhood, the limit of firm 1 's expected profit's can be obtained as:

$$\lim_{P \to U_{1T}-} W_1^2(p;q,T) = q_0 \delta \bar{W}_2^2(q,T-1) + q_1 \delta V(\bar{P},1,T-1) + \sum_{i=2}^{+\infty} q_i U_{1T}. \quad (68)$$

Considering (9), (10) and our assumption $p_{N,T}^{*} < I_{1T} = I_{2T}$, firm 1's expected profit for I_{1T} is more than for a right neighborhood of U_{1T} :

$$\lim_{p \to U_{1T}-} W_1^2(p;q,T) \leq q_0 \delta \bar{W}_2^2(q,T-1) + (1-q_0) P_{N,T}^{*} < q_0 \bar{W}_2^2(q,T-1) + (1-q_0) I_{1T}. \quad (69)$$

Based on limit definition, there exists $\epsilon_1 > 0$ such that for $p \in (U_{1T} - \varepsilon_1, U_{1T})$, the expected profit is less than the expected profit for I_{1T}. So, firm 1 never assigns a positive probability to this interval in equilibrium, and it is a contradiction. Thus, the following equation is proved:

$$l_{1T} = I_{2T} = P_{2,T}^{*}. \quad (70)$$

Now, we examine the second part of lemma 3.6. First, assume $U_{1T} \neq U_{2T}$ and $U_{1T} < U_{2T}$. According to the previous section, we have $p_{2,T}^{*} \leq U_{1T} < U_{2T}$. So, we have:

$$\exists \eta \in (U_{1T}, U_{2T}) \mid 0 < F_{2T}(\eta) < 1. \quad (71)$$

Firm 2's expected profit can be calculated from the following equation for $p \in (U_{1T}, U_{2T}]$:

$$W_2^2(p;q,T) = q_0 \delta V_1(T-1) + q_1 \delta \bar{W}_2^2(q,T-1) + \sum_{l=2}^{+\infty} q_i p < q_0 \delta V_1(T-1) + q_1 \delta V(\bar{P},1,T-1) + \sum_{l=2}^{+\infty} q_l U_{2T} \quad (72)$$

The right-hand side of (70) is the expected profit at U_{2T}. As a result, since firm 1 is absent in $p \in (U_{1T}, U_{2T}]$, firm 2 assigns no probability to (U_{1T}, U_{2T}). In addition, because we know U_{2T} is greater than U_{1T}, firm 2 has to assigns positive probability to the upper bound U_{2T}. In addition, it is not possible to have equilibrium if firm 2 assigns a positive probability to U_{1T}. To see this, we use proof by contradiction. Thus, assuming $F_{2T}(U_{1T}) - F_{2T}(U_{1T-}) > 0$, firm 1's expected profit for U_{1T} is less than its value for a right neighborhood of U_{1T}. This is due the fact that we have $V(\bar{P},1,T-1) < P_{N,T}^{*} < U_{1T}$. So, firm 1 assigns zero probability to U_{1T}. Considering all we've mentioned, firm 2's expected profit of U_{1T} would be less than its corresponding value for U_{2T}. So, we showed that firm 2 assigns zero probability to U_{1T}. Let's examine the situation this time from the standpoint of firm 1 . Because firm 2 is absent in (U_{1T}, U_{2T}), we have:

$$W_1^2(U_{1T};q,T)=\lim_{P\to U_{1T}-}W_1^2(p;q,T)=q_0\delta\bar{W}_1^2(q,T-1)+q_1[F_{2T}(U_{1T})\delta V(\bar{P},1,T-1)$$
$$+\left(1-F_{2T}(U_{1T})\right)U_{1T}]+\sum_{i=2}^{+\infty}q_iU_{1T}<q_0\delta\bar{W}_1^2(q,T-1)+q_1[F_{2T}(U_{1T})\delta V(\bar{P},1,T-1)$$
$$+\left(1-F_{2T}(U_{1T})\right)\eta]+\sum_{i=2}^{+\infty}q_i\eta \tag{73}$$

when $\eta\in(U_{1T},U_{2T})$. The right-hand side of (72) demonstrates the expected profit of η. Thus, we showed that there is a pure strategy for firm 1 for which the expected profit is greater than the expected profit of the prices in a left neighborhood of U_{1T}. Based on Theorem 4.1, firm 1 should assign zero probability to an interval $(U_{1T}-\epsilon,U_{1T}\ ]$, when ϵ is small positive value. This is a contradiction, and we have $U_{1T}=U_{2T}$ in equilibrium. In order to prove that $U_{1T}=U_{2T}<\bar{P}$ is not feasible, we employ proof by contradiction. We assume $U_{1T}=U_{2T}<\bar{P}$ in equilibrium. According to this assumption, there are two probable states that we examine here.

1. Assume $1-F_{1T}(U_{1T-})>0$. The limit of the expected profit of firm 2 , when p approaches to U_{1T} from its left neighbourhood is calculated as follows:

$$\lim_{P\to U_{1T}-}W_2^2(p;q,T)=q_0\delta\bar{W}_2^2(q,T-1)+q_1\left[F_{1T}\left(U_{1T\_}\right)\delta V(\bar{P},1,T-1)\right.$$
$$\left.+\left(1-F_{1T}\left(U_{1T\_}\right)\right)U_{1T}\right]+\sum_{i=2}^{+\infty}q_iU_{1T} \tag{74}$$

A. 7 Proof of lemma 3.7.

If the domain of two firms' strategies is $\left[P_{2,T}^*,\bar{P}\right]$, both firms assign zero probability to $P_{2,T}^*$. To show this, we can employ proof by contradiction. Assume firm 2 assigns a positive probability to $P_{2,T}^*$. Then, we have:

$$\lim_{p\to P_{2,T}-}W_2^2(p;q,T)=q_0\delta\bar{W}_i^2(q,T-1)+q_1P_{2,T}^*+\sum_{i=2}^{+\infty}q_iP_{2,T}^*,$$
$$\lim_{p\to P_{2,T}+}W_2^2(p;q,T)=W_2^2\left(P_{2,T}^*;q,T\right)<q_0\delta\bar{W}_i^2(q,T-1)+q_1P_{2,T}^*+\sum_{T=2}^{+\infty}q_iP_{2T}^*. \tag{75}$$

We have (75), due to the fact that $P_{2,T}^*>\delta V(\bar{P},1,T-1)$. As a result, based on limit definition, there are $\epsilon_1>0$ and $\epsilon_2>0$ such that for $p\in\left[P_{2,T}^*,P_{2,T}^*+\varepsilon_1\right)$, the expected profit is less than the expected profit for $p\in\left(P_{2,T}^*-\epsilon_2,P_{2T}^*\right)$. This is a contradiction with our assumption regarding $P_{2,T}^*$ being the lower bound. So, both firms assign a probability equal to zero to $P_{2,T}^*$. Based on Theorem 4.1, the expected profit of a set of pure strategies cannot be greater than the equilibrium expected profit. We also have that the expected profit of a set of pure strategies with positive assigned probability should not be less than the equilibrium expected profit. Furthermore, we know that there $\forall\epsilon>0$, we have $F_{1T}(P_{2T}^*+\epsilon)>0$. For a small ϵ, we assume the expected profit of the set of points in $(P_{2T}^*,P_{2T}^*+\epsilon)$ is equal to a common value W. Since we have:

$$W=\lim_{p\to P_{2,T}^*+}W_2^2(p;q,T)=q_0\delta\bar{W}_i^2(q,T-1)+q_1P_{2,T}^*+\sum_{T=2}^{+\infty}q_iP_{2T}^*, \tag{76}$$

and the initial condition $\bar{W}_i^2(q,1)=V(\bar{P},2,1)=0$, we can conclude from (10), that the expected profit in equilibrium is $V(\bar{P},2,T)$, which is similar for both firms. To complete the proof, we again use proof by contradiction. Let us assume that the equilibrium strategy is not symmetric, then we have:

$$\exists p\in(P_{2T}^*,\bar{P})\to F_{2T}(p)>F_{1T}(p). \tag{77}$$

Because cumulative distribution function is right-continues, for any small parameters $\epsilon_1,\epsilon_2>0$ there exists η such that:

$$F_{iT}(\eta)-F_{iT}(\eta^-)=0$$
$$F_{iT}(\eta)-F_{iT}(p)=e_i<\epsilon_i. \tag{78}$$

Then, firm 1's and firm 2's expected profit at $p = \eta$ can be calculated from the following equations:

$$W_1^2(\eta; q, T) = q_0\delta\bar{W}_1^2(q, T-1) + q_1[(F_{2T}(p) + e_2)\delta V(\bar{P}, 1, T-1) + \big(1 - (F_{2T}(p) + e_2)\big)\eta] + \sum_{i=2}^{+\infty} q_i\eta \tag{79}$$

$$W_2^2(\eta; q, T) = q_0\delta\bar{W}_2^2(q, T-1) + q_1[(F_{1T}(p) + e_1)\delta V(\bar{P}, 1, T-1) + \big(1 - (F_{1T}(p) + e_1)\big)\eta] + \sum_{i=2}^{+\infty} q_i\eta \tag{80}$$

Considering negligible ϵ_1, ϵ_2, firm 1's expected profit is greater than firm 2's in η. This is in contradiction with our previous result of similar equilibrium profit. So, the assumption of different equilibrium distribution function cannot be true, and the result is established.

A. 8 Proof of Proposition 3.8. This can be shown regarding the result of Proposition 3.5 and Lemma 3.8.

A. 9 Proof of Proposition 3.9. To show that is a mixed-strategy equilibrium, we have to show that it satisfies the necessary and sufficient conditions in Theorem 4.1. In addition, we need to show that it is a cumulative distribution function, so it should be a monotonically increasing function on $[0,1]$. We have the following equation according to the definition:

$$V(\bar{P}, N, 0) = 0. \tag{81}$$

In addition, if we assume that all the firms have a common mixed-strategy $F(p)$, the expected profit of firm 1 can be calculated as follows:

$$W_1^N(p; q, 1) = q_0\delta \bar{W_1^N}(q, 0\big) + \sum_{i=1}^{N-1} q_i Z_{i-1,N}(F(p))p + \sum_{i=N}^{+\infty} q_i p \tag{82}$$

Now, we obtain the equilibrium strategy which results in $W_1^N(p; q, 1) = V(\bar{P}, N, 1)$. Thus, by substituting $W_1^N(p; q, 1) = V(\bar{P}, N, 1)$ and $\bar{W_1^N}(q, 0\big) = V(\bar{P}, N, 0) = 0$ in (80), we can conclude the following equation:

$$\sum_{i=1}^{N-1} q_i Z_{i-1,N}(F(p))p + \sum_{l=N}^{+\infty} q_i p = \sum_{i=n}^{+\infty} q_i\bar{p}. \tag{83}$$

Thus, if we solve (82) for $F(p)$, we have:

$$F(p) = \begin{cases} 0 & p \le P_{N,T}^* \\ G^{-1}(p) & P_{N,T}^* < p < \bar{p} \\ 1 & p \ge \bar{p} \end{cases}, \quad G(x) = \frac{\sum_{i=n}^{+\infty} q_i\bar{p}}{\sum_{i=n}^{+\infty} q_i + \sum_{i=1}^{n-1} q_i Z_{i-1}(x)}. \tag{84}$$

Since we have $W_{N,1}(p) = F(p)$ in (83), all three equilibrium conditions are satisfied. Because the assumption of inductive reasoning is proved, we can assume that we have the equilibrium profit for $T-1$. As a result, $W_1^N(p; q, T)$ can be calculated as follows:

$$W_1^N(p; q, T) = q_0 V(\bar{P}, N, T-1) + \sum_{i=1}^{N-1} q_i\big[Z_{i-1,N}(F(p))p + \big(1 - Z_{i-1,N}(F(p)\big) \delta V(\bar{p}, N-i, T-1)] + \sum_{i=N}^{+\infty} q_i p = \sum_{i=0}^{N-1} q_i \delta V(\bar{p}, N-i, T-1) + \sum_{i=N}^{+\infty} q_i\bar{p}. \tag{85}$$

If we solve (84) for $F(p)$, we have $F(p) = W_{N,T}(p)$. So, the equilibrium conditions are satisfied, and the result is established.

A.10 Proof of lemma 3.10.

To show that $W_{N,T}(p)$ is a function, we first should show that its inverse function is an injective function. Now we show that its inverse function is monotonous and increasing at $p \in \left(P^*_{N,T}\bar{p}\right)$. The denominator and numerator of the function is positive and monotonous, so because $Z_{K,N}(x)$ is a continues and strictly decreasing this function is monotonous. The derivative of these functions can be calculated from the following relations:

$$\frac{dG(x)}{dx} = \sum_{i=1}^{n-1} q_i^2 (\delta V(\bar{p}, N-i, T-1) - \bar{P}) \frac{dZ_{i-1}(x)}{dx} \tag{86}$$
$$\forall x \in (0,1) \rightarrow \frac{dZ_k(x)}{dx} = -(k+1)\binom{n-1}{k+1}(1-x)^{n-2-k}x^k < 0$$

So, $W_{N,T}(p)$ is an increasing monotonous function at $p \in \left(P^*_{N,T}, \bar{p}\right)$.

A. 11 Proof of Proposition 3.11.

Assuming (23), the probability every firm assigns to lower bound is zero. This can be shown by employing a proof by contradiction. If a firm 2 assigns positive probability to this price, firm 1's expected profit when p approaches to $P^*_{N,T}$ from its left neighbourhood is more than its expected profit at $P^*_{N,T}$ and more than its expected profit when p approaches to $P^*_{N,T}$ from its right-neighbourhood. This is in contradiction with equilibrium conditions. Here, we have the limit of the expected profit of firm 1 as follows:

$$\lim_{P \to p^*_{N,T}-} W_1^N(p; q, T) = \delta \bar{W}_1^N(q, T-1) + \sum_{i=1}^{\infty} q_t p^*_{N,T}. \tag{87}$$

Since $\sum_{i=2}^{+\infty} q_i > 0$, we have $V(\bar{P}, N-i, T-1) < P^*_{N,T}$. As a result, when at least one of other $N-1$ firms assign a positive probability to $P^*_{N,T}$, we have the following equation at $p = P^*_{N,T}$:

$$\lim_{p \to P^*_{N,T}+} W_1^N(p; q, T) = W_1^N(p; q, T) < \lim_{P \to p^*_{N,T}-} W_1^N(p; q, T) \tag{88}$$

Based on (86) and (87) and limit definition, we can find two intervals at the left and right of $p^*_{N,T}$, as the expected profit of the points in the left interval is greater than the expected profit of the points in the right interval containing $p^*_{N,T}$. So it is in contradiction with equilibrium definition and firm 1 and firm 2 assign zero probability to $p^*_{N,T}$ and the expected profit is $V(\bar{P}, N, T)$. (It can be shown with the same reasoning as the one we employed for duopoly)
If we assume that the equilibrium strategy is asymmetric, we can find p as the following equation holds:

$$F_{-1NT}(p) > F_{-2NT}(p). \tag{89}$$

Because cumulative distribution function is right-continues, ϵ and η can be found as for $i = 1{,}2$ the equations in (90) hold:

$$\begin{gathered} F_{-iNT}(\eta^-) - F_{-iNT}(\eta) = 0, \\ F_{-iNT}(\eta) - F_{-iNT}(p) = e_i < \epsilon_i. \end{gathered} \tag{90}$$

The existence of p can be shown from the fact that we assume the random strategies are independent, so the we have $F_{1NT}(p) = 1 - \prod_{i=2}^{N}(1 - F_{iNT}(p))$. So, different values for F_{iNT} and F_{jNT} would result in different values for F_{-iNT} and F_{-jNT}. Based on limit definition, ϵ_i can be chosen as small as we wish, and firm 1 's expected profit can be obtained from the following equation:

$$W_1^N(p; q, T) = \delta \bar{W}_1^N(q, T-1) + \sum_{i=1}^{N-1} q_i \left[(F_{-1NT}(p) + e_1) \delta V(\bar{P}, N-i, T-1) + \left(1 - (F_{-1NT}(p) + e_1)\right)\eta \right] + \sum_{i=2}^{+\infty} q_i \eta \tag{91}$$

for $p \in \left(p, p + \min(\epsilon_1, \epsilon_2)\right)$. Similar equation can be obtained for firm 2. Considering a small enough ϵ_i, firm 1's expected profit is less than firm 2's and it is in contradiction with our assumption of identical expected profit. Thus, the assumption of different equilibrium distribution function is not true.