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Near Exact Calculation of American Options

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Abstract

A new practical approach for the analysis of American (bond) options is developed which is a combination of the closed form solutions and binomial lattice models. The model is calibrated to the observed term structure of rates and traded volatilities and is arbitrage free. The convergence is very fast, but numerically intensive. By extrapolation the near exact premium of an American (bond) option can be calculated.

Keywords: American option, bond option, callable bond, option, exercise boundary

1. Introduction

An American option gives the option holder the right to exercise the option at any time during its exercise period, unlike a European option which can only be exercised at expiration. For an asset that doesn't pay any dividend or has no accrued income, an American call option is like a European option. It is never optimal to exercise the option early and pay for the underlying asset that pays no dividend with a capital that could earn short term rate until the expiration of the option. Additionally, there is a chance that the asset could sell off below the exercise price by the expiration of the option. On the other hand, a put for such an asset can be exercised early if the reward of receiving short term rate on the proceeds of the put outweighs the risks of the underlying asset rising above the strike price by the expiration of the option.

Analysis of American options for assets that pay dividend, in particular bond options, is significantly more complicated than European options. The difference between short term and long term rates is one of the driving forces for the early exercise of an American bond option. During a steep yield curve environment, a call option is more likely to be exercised early in order to receive long term rates implied by the underlying asset, instead of investing the capital at short term rate and vice versa for a put option. This dynamic reverses in inverted yield curve environment. Additionally, a bond is a varying asset as a function of time; its price has a maximum value and its risks and interest rate sensitivity fall as it approaches its maturity date.

In general, models for pricing interest rate options can be classified as short rate and market implied rates models. Short rate models, in particular Hull and White (1990) and Black & Karasinsky (1991) appear to be the most prevalent models in the marketplace. These models use lognormal distribution of rates. Variants of the short rate models such as Ho-Lee (1986) which uses normal distribution for rates are not as desirable, even though interest rate paths can be calibrated to price the entire yield curve exactly. The market based models are calibrated in such a way to reproduce the observed term structure of rates and traded volatilities. Among them Black, Derman and Toy (1990) and the more general model of Heath, Jarrow and Morton (HJM) (1990) which can be formulated using normal or lognormal distribution of rates.

For an American option, the option price and the exercise boundary have to be calculated at the same time and for this reason there are no exact or closed form solutions for them. Many analytical approaches to American options are not convergent or their convergence is questionable. For example, the model of Barone-Adesi and Whaley (1987) who use the quadratic model of MacMillan (1986) is not convergent, neither are the subsequent extensions (See Ju and Zhong (1999)). Likewise, Sullivan (2000) approximation using Chebyshev polynomials that employs discrete exercise dates, has unknown convergence.

Most of the analytical methods for the American bond options have focused on the fixed maturity zero coupon bond options at the time of exercise. Alobaidi and Mallier (2017) use Vasicek (1980) interest rate model to derive an expression for the behavior of the exercise boundary close to the expiration of American options. Unlike most traded bond options and callable bonds, where the maturity of the underlying bond is known in advance, such options have little practical application.

The biggest effort in pricing bond options has been on the development of numerical methods for their analysis. Among the first models, is Brennan and Schwartz (1979) numerical partial differential equation scheme. The binomial lattice model was first proposed by Sharpe (1978) and formalized by Cox, Ross and Rubinstein (1979). It is simple and straightforward to implement and is used widely for pricing American or Bermudan options.

Among the most popular methods of valuing risk free American options such as swaptions is the binomial model. The convergence rate of binomial model which is of the order of $\sqrt{N}$ is very slow and at the early stages of the tree, it is too sparsely populated to accurately account for the early exercise opportunities. This is in particular true for call swaptions in a steep yield curve where the forward coupon rate is higher than the spot coupon rate.

Many other models for the valuation of bond options, such as the popular Black 76 (1976), which is a rate implementation of the standard Black and Scholes (1973) model can't be used for American options. Simozar (2019) argues that Black-76 model is not arbitrage free and fails to adjust the discount yield which itself is a function of the exercise yield of the option. Xie (2009) has developed a numerical method for the calculation of the optimum exercise of American options using Vasicek (1977) model. However, calibration of the model to replicate the observed term structure of rates and volatilities may not be possible. Additionally, Vasicek (1977) model can potentially lead to negative interest rates which is not desirable.

The popularity of binomial models for the analysis of American bond options is due to their flexibility and robustness. Binomial models can be calibrated to replicate forward rates, observed volatilities and be arbitrage free at the same time. The discount rate at every node of a binomial interest rate tree, is the then prevailing rate at that node. Barone-Adesi (2005) has a detailed review of the numerical methods for the calculation of American options. A significant portion of methods are for put options on non-dividend paying stocks. Literature on callable bonds, in particular bonds that have multiple strike prices, such as bonds that are issued in the high yield market, is very light and sparse.

Our approach for the calculation of American options is a combination of the closed form solutions and binomial model It has the advantages of the binomial models and the accuracy of closed form solutions, if one existed. It always converges, can be calibrated to market observed volatilities and can handle varying strike prices. We create layers in the exercise period of the option and at each layer we calculate a forward distribution of rates that matches the observed market volatility and implied forward rate for that expiration and maturity. This is how the model is calibrated to the market observed volatilities and term structure of rates. Next, we map out the exercise boundary by backward induction and discount the expected exercised prices at the appropriate discount rate. In short, our model is arbitrage free, convergent, very accurate and highly suitable for real market applications.

Given the complex nature of American bond options, we looked for a widely available market benchmark to gauge the effectiveness of our model compared to the existing ones. The difference between the premium calculations of our method and those of Bloomberg's as a representative of the market models, can be of the order of 50% or more.

2. Probability Distribution

Our model for American Options has the accuracy of a closed form solution if one existed and uses a manageable number of steps. We assume that there exist a Term Structure of Rates (TSR) and Term Structure of Volatility (TSV) surface that are both continuous and infinitely differentiable. For non-interest rate options such as stocks or commodities, we assume that the term structure of the volatility surface (as a function of the forward time and expiration) and the TSR are known.

Similar to the binomial model, we use multiple layers to evaluate option prices in each layer. Unlike binomial model, we use the same number of points in every layer for the analysis of the option. For example, for an option that can be exercised between 5 and 10 years from now, we create 21 layers at quarterly intervals and evaluate the option only at those layers.

The first step in analyzing American options is to calculate the arbitrage free probability distribution of rates or asset prices at all layers. For bonds, we first calculate the price of a risk-free bond with cash flows c_i, at time t_i with a yield y_i as

$$p_t = \sum_i c_i e^{-y_i t_i} \tag{1}$$

We assume that forward bond yields (y) in (1) or the forward asset price for non-interest rate securities follow a geometric Brownian motion (GBM) with time dependent drift, i.e.,

$$\begin{aligned} d\ell n(y) &= \mu_y(t)dt + \upsilon_y(t)dB_y(t) \\ d\ell n(s) &= \mu_s(t)dt + \upsilon_s(t)dB_s(t) \end{aligned} \tag{2}$$

Where μ_y is the drift, υ_y is the volatility and B_y is a Brownian motion for an interest rate (yield) process. The same parameters for a non-interest rate process is denoted by subscript *s*. By applying Ito's lemma and solving the resulting Black Scholes PDE or simply noting that (2) represents a GBM whose solution is a Geometric Normal Distribution (GND), we find the probability density function ψ as the solution to the diffusion equation as

$$\begin{aligned} d\psi_f(y,t) &= Ae^{-\frac{\left(\ell n(y)-\ell n(y_f)-\mu\right)^2}{2\sigma^2}} d\ell n\left(y\right) \\ \sigma(t)^2 &= \upsilon(t)^2 t \end{aligned} \tag{3}$$

For bonds, there are two boundary conditions that need to be met. The first one, the normalization factor, i.e., requiring that the sum of all probabilities is equal to one, namely,

$$\begin{aligned} 1 &= A\int_{y=0}^{\infty} e^{-\frac{\left(\ell n(y)-\ell n(y_f)-\mu\right)^2}{2\sigma^2}} d\ell n\left(y\right) \\ A &= \frac{1}{\sqrt{2\pi\sigma}} \end{aligned} \tag{4}$$

The second boundary condition is the arbitrage free condition which requires that the expected value of the forward price over all probabilities be equal to the forward price; this establishes the drift parameter μ for a given time *t*. Thus,

$$\begin{aligned} p_{f,t}(y_f) &= \int p_{f,t}(y)d\psi_f(y,t) \\ p_{f,t} &= \tfrac{1}{\sqrt{2\pi\sigma}}\int_{y=0}^{\infty} p(y)e^{-\frac{\left(\ell n(y)-\ell n(y_f)-\mu\right)^2}{2\sigma^2}} d\ell n\left(y\right) \end{aligned} \tag{5}$$

Where p_f is the forward price of the bond at the forward yield y_f. For equities or commodities where the primary driver of the forward distribution function is price, the value of drift is $\mu = -\frac{1}{2}\sigma^2$. However, for bonds the distribution function is a function of yield while the arbitrage free requirement is on price and since price-yield relationship is not linear, therefore $\mu \neq -\frac{1}{2}\sigma^2$. For short term bonds, where the price-yield relationship is almost linear, $\mu \approx -\frac{1}{2}\sigma^2$

In practice, it is easier to work with a Unit Normal Distribution (UND) than with a GND by making a simple transformation. Thus, for bonds we can write,

$$\begin{aligned} u &= \frac{\ell n(y) - \ell n(y_f) - \mu}{\sigma} \\ y &= y_f \exp(\sigma u + \mu) \end{aligned} \tag{6}$$

With this transformation, the arbitrage free requirement (5), will become,

$$p_f = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} p(y)e^{-\frac{u^2}{2}}du \tag{7}$$

3. Forward Distribution In Binomial Model

The yield distribution of a bond at time t_1 from a starting yield of y_0 at time t_0 and probability $f_0 = 1$, denoted by (t_0, y_0, f_0) is governed by (5) To draw parallels between our model and the binomial model, assume that (t_0, y_0, f_0) can evolve into only the following two states at time t_1. (See Figure 1)

$$(t_0, y_0, f_0) \rightarrow \begin{cases} (t_1, y_{1,a}, f_{1,a}) \\ (t_1, y_{1,b}, f_{1,b}) \end{cases} \tag{8}$$

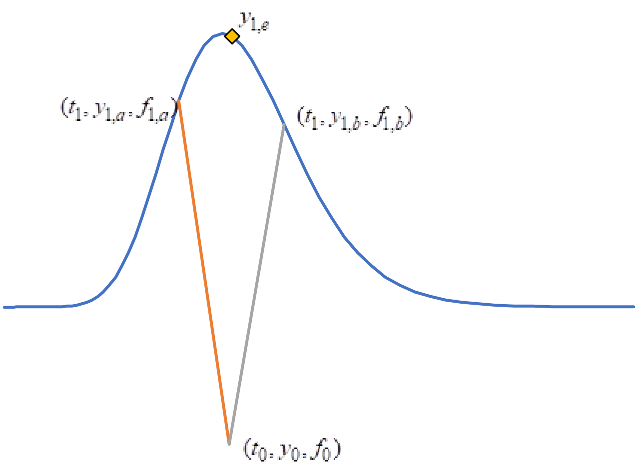

Figure 1. Probability distribution density in GBM

Where $y_{1,e}$ is the equilibrium forward yield that can be calculated from the TSR. Every forward yield $y_{1,j}$ will have an associated forward price $p_{1,j}$. Our normalization equation for the binomial will be

$$f_{1,a} + f_{1,b} = 1 \tag{9}$$

If the yield volatility at time t_1 is v_1, then points a and b should be selected in such a way to get the expected volatility, i.e.,

$$f_{1,a}\left(\ell n(\frac{y_{1,a}}{y_{1,e}})\right)^2 + f_{1,b}\left(\ell n(\frac{y_{1,a}}{y_{1,e}})\right)^2 = v_1^2 \tag{10}$$

Next, we need to impose the arbitrage free requirement, namely,

$$f_{1,a}p_{1,a} + f_{1,b}p_{1,b} = p_{1,e} \tag{11}$$

Thus, in the binomial model, there are 3 constraints to be met, but there are 4 variables, namely, $f_{1,a}$, $f_{1,b}$, $y_{1,a}$ and $y_{1,b}$. For this reason, there are many flavors of the binomial model. One of the popular modes is to assume equal probability for points *a* and *b*, i.e., $f_{1,a} = f_{1,b}$. Another popular method is to assume that the equilibrium yield $y_{1,e}$ is the geometric mean of the yields at *a* and *b*, i.e., $y_{1,e}^2 = y_{1,a}y_{1,b}$. Any other combination that meet the above 3 equations can be used and at the limit, they all converge to the continuous model of the distribution function. See Josh (2008) or Chance (2008).

In our approach, we take the continuous distribution of forward rates or asset prices and break it up into many buckets of equal width in the UND space. This is the exact method that is used in the numerical calculation of the distribution function using Simpson's rule. However, we keep the buckets discrete. The weight or probability of each bucket is equal to its area or average height or density and we assume that all the weight of the bucket is in its middle. Thus, from our initial position of (t_0, y_0, f_0) we will have *n* possible paths leading to $(t_1, y_{1,i}, f_{1,i})$ points (*i=1-n*) instead of just 2 paths. This is how the first layer is constructed.

4. Multinomial Tree

The next step is to evolve the distribution from t_1 to t_2 using the interlayer volatility. This step requires the calculation of the interlayer drift and volatility.

Consider a non-time varying asset such as a stock or a commodity. If the expected standard deviation of its price and drift at forward times t_1 and t_2 are (σ_1, μ_1) and (σ_2, μ_2) and in the interval $t_2 - t_1$ they are (σ_{12}, μ_{12}), we have the following relationships (See Simozar 2019)

$$\begin{aligned} \sigma_1^2 + \sigma_{12}^2 &= \sigma_2^2 \\ \mu_1 + \mu_{12} &= \mu_2 \end{aligned} \tag{12}$$

Thus, we can calculate the interlayer drift and standard deviations for non-varying assets. If we use the drift and standard deviation of the forward bond at t_2 for calculation at time t_1, then the above equation would be valid for bonds as well. However, the forward bonds at t_1 and t_2 are slightly different instruments and the above equation will be only an approximation. Considering that we generally use interval spacings that are small compared to the maturity of the bond, the volatility of the forward bonds at t_1 and t_2 will be very close and the above approximation would be very reasonable. The same approximation is made in calculating the forward step volatility in the binomial option pricing model.

We use the same number of buckets in layer 2. Since the size of buckets in yield space is proportional to the standard deviation of the distribution, they will be larger than the buckets in layer 1, however, in UND space, all bucket sizes will be the same. Each bucket in layer 1 is evolved into all the buckets in layer 2, by assuming that its weight is in the middle of the bucket plus adjustments (See Appendix). See Figure 2.

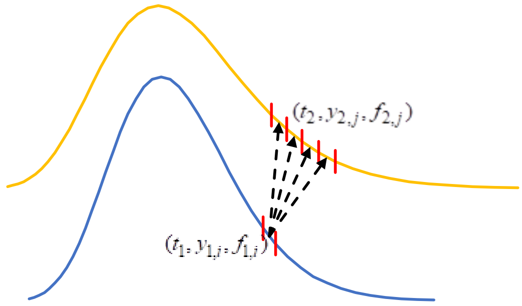

Figure 2. Probability distribution density in GBM

The bucket in layer 1 at $(t_1, y_{1,i}, f_{1,i})$ is transformed to buckets in layer 2 $(t_2, y_{2,j}, f_{2,j})$ using (σ_{12}, μ_{12}) parameters. However, the buckets in layer 2 have bucket sizes and distributions governed by (σ_2, μ_2). Therefore, we must find the fraction of bucket $(t_1, y_{1,i}, f_{1,i})$ that falls into each bucket $(t_2, y_{2,j}, f_{2,j})$ using (σ_2, μ_2) distribution.

In a normal or lognormal distribution, the center of mass of each bucket is a little closer to the center of the distribution than the middle point of the bucket that we use. Using the middle point, will cause the distribution to spread out gradually over many layers. After about 25 layers, the central point of the distribution loses about 2% of its height due to this assumption. This can have a small impact on the price of an option and adjustment to this approximation is necessary for very accurate calculation of option prices. Without the adjustment, the methodology will still produce significantly more accurate calculation than binomial model.

5. Exercise Boundary

The next step is to calculate the exercise boundary at all times during the exercise period of the option. Exercise boundary is the point on the map of future rates that the prices of exercised and unexercised options are equal. For example, for call options, the American option is exercised immediately, if the rate is below the exercise boundary. We calculate the arbitrage free distribution of rates at the expiration date, just as we would a European option. At expiration date of the American option or just before maturity of a bond that is callable to maturity, an American option is like a European option; if it is in the money, it will be exercised in this layer *n*. For this layer we need to calculate the forward price of the underlying asset at regular intervals and use numerical integration (Simpson’s Rule) to calculate the option value at forward exercise prices.

Next, we analyze layer *n-1* by calculating the forward arbitrage free probability distribution function (5). For every point in layer *n-1*, we calculate the exercised and unexercised prices of the option. The unexercised price is calculated between layers *n-1* and *n*, knowing interlayer volatility and drift (12). Each point in layer *n-1* is evolved to a distribution in layer *n*. Since we know the option price at every point in layer *n*, we can discount it by the appropriate interlayer discount function using interlayer probability distribution. Not all forward points in this layer that are in the money and thus in the exercise zone, are economical to exercise. If the unexercised option value at a given point, in the exercise zone is lower than the benefit of exercising, then it is economical to exercise early. For a lattice of *m* points (*m-1* intervals or buckets), it requires about m^2 operations. Once the exercised and unexercised prices are calculated for layer *n-1*, the exercise boundary can be calculated. It should be noted that the exercised price needs to be calculated only if it is positive, but the unexercised price is positive everywhere before expiration date. This process is repeated for every layer until the exercise boundary is fully mapped. For bonds, calls are more economical to be exercised on the left side (lower rates) of the exercise boundary and puts on the right side. However, there is no need to calculate the security price at all these points. We can use quadratic interpolation to calculate the forward price in layer *n*.

The most time consuming part of the calculation is usually the calculation of the security price which needs to be calculated at each node of each layer only once. Quadratic interpolation can then be used to calculate it at every other point. Thus, the security price calculation that is used for drift calculation can be used for building the multinomial trees without any further adjustment.

Given that at the exercise boundary the prices of exercised and unexercised options are the same, deviations in interest rates or security prices in the interlayer intervals will result in small errors. We will review this point in more detail later in this paper.

For options with a single strike price or strike yield, the exercise boundary is a continuous function of time. However, for an option with multiple strike prices such as a callable bond that has different call prices at different times, the exercise boundary is discontinuous. If the exercise boundary is discontinuous, there are layers where no exercise is possible.

Figure 3 shows the exercise boundary of an American Swaption with expiration of 5 years and maturity at expiration of 10 years. The left axis is the number of standard deviations in the UND space that defines the exercise boundary and the right axis is the equivalent yield. The equilibrium forward yield is about 2% and therefore the boundary's yield approaches 2% at time to expiration approaches zero. The slope of the exercise boundary increases as it approaches zero time to maturity and is consistent with the findings of Evan, Kuske and Keller (2002).

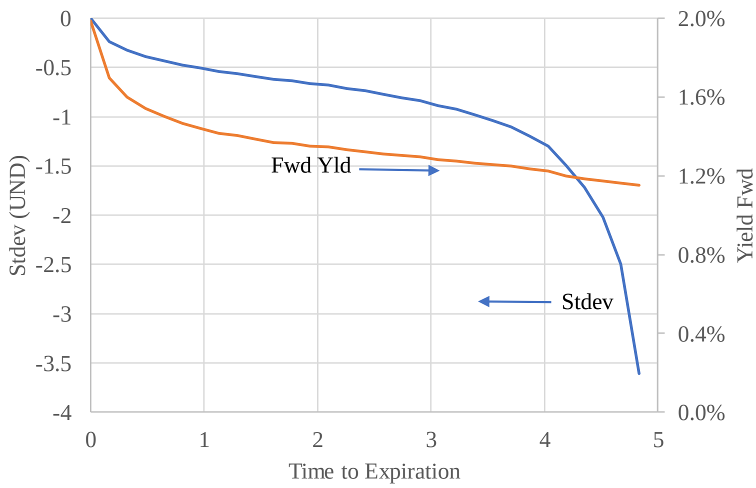

Figure 3. Exercise boundary of 5x10 Swaption

Callable bonds are usually different from American Receiver Swaptions, in that there is no expiration date and the option is a wasting option if not exercised. For this reason, the incentive to exercise early is significantly higher than for swaptions. Some bond options have call schedules, where the call price at earlier dates is higher than at later dates. For bonds with a call schedule, the exercise boundary is discontinuous and there are regions where it is not economical to exercise at any yield. For example, for a bond that has an exercise price of 102 which falls to 100 in a month, no exercise yield is economical, unless the coupon rate of the bond is at least 24%. We will analyze a bond with the following characteristics:

Coupon	Maturity	Call 1	Call 2	Call 3	Price
8.375	1/27/28	7/18/23	7/18/24	7/18/25	102.7
		104.1875	102.09375	100	

Assuming that the volatility of this bond is perfectly correlated with the volatility of Swaptions, we can map out the exercise boundary of this bond.

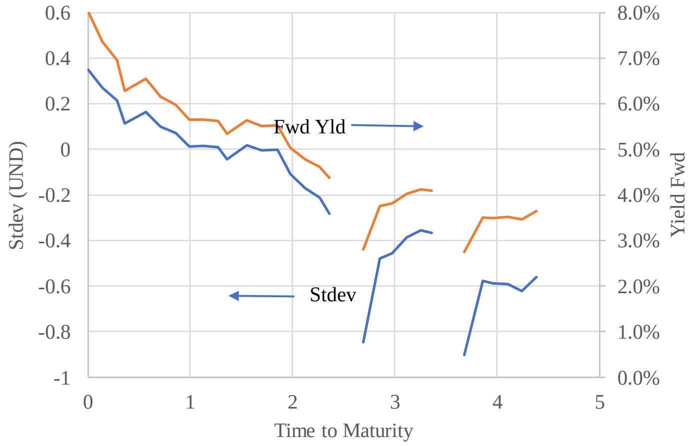

Figure 4. Exercise boundary of a Corporate bond with Call Schedule

Other than the discontinuities which happen just before exercise price changes, the curves are not smooth like Figure 3 due to coupon effects. The risk profile of a bond changes just before and after coupon payment. For example, the duration has a jump increase immediately after a coupon payment and starts declining as it approaches the maturity date, resulting in a saw tooth pattern. Similar pattern exists for Swaptions as well, however, the coupon rate is much lower and time to maturity is significantly higher and it appears smooth to the eye. Given that the bond trades at a premium, there is a large incentive to call the bond early. In Figure 4, the exercise boundary in the range of 1-2 years is at about zero Stdev.

6. Feed Forward Pricing

The next step in evaluating American options is to work from the first time that the option is exercisable and take out all the paths that have led to early exercise and evolve unexercised paths to forward dates.

In order to calculate the option price, we start with the first layer, i.e., the first time the American bond option can be exercised. The buckets that are in the exercise zone are exercised and the present value is calculated based on the discount function and their weight. The contribution of these buckets are then taken out of the distribution. The remainder of the buckets are evolved to the next layer and the fractions that fall into the exercise zone are exercised. The buckets that are exercisable are calculated assuming that the exercise point is in the middle of the bucket. We will then make adjustments to the option price considering that the exercise point is continuously spread in each bucket. (See Appendix). The present value of each exercisable bucket is calculated using the exercise dependent discount function. Since the exercise boundary usually falls somewhere in a bucket, the contribution of the bucket at the exercise boundary to the price requires special consideration.

Table 1. Calculated Swaptions prices and comparison with Bloomberg values – January 3, 2020

C/P	A/E	Expire	Vol	Fwd Rate	Fwd BBG*	Prem	Prem No Adj	Prem BBG*
C	E	4/3/20	0.3496	1.7594	1.7574	1.1126	1.1124	1.2241
P	E	4/3/20	0.3496	1.7594	1.7574	1.1116	1.1118	1.2513
C	A	4/3/20	0.3496	1.7594	1.7574	1.1163	1.1162	3.9283
P	A	4/3/20	0.3496	1.7594	1.7574	1.1151	1.1152	4.6450
C	E	1/4/21	0.3406	1.7898	1.7878	2.1715	2.1656	2.3767
P	E	1/4/21	0.3406	1.7898	1.7878	2.1551	2.1646	2.3767
C	A	1/4/21	0.3406	1.7898	1.7878	2.2565	2.2540	4.3306
P	A	1/4/21	0.3406	1.7898	1.7878	2.2152	2.2195	5.0792
C	E	1/3/25	0.3021	1.9982	2.0029	4.4175	4.3036	4.8212
P	E	1/3/25	0.3021	1.9982	2.0029	3.9796	4.3013	4.8212
C	A	1/3/25	0.3021	1.9982	2.0029	5.7188	5.6871	5.9574
P	A	1/3/25	0.3021	1.9982	2.0029	4.7851	4.8971	6.4101

* Source: Bloomberg Finance L. P.

Table 1 lists at-the-money (ATM) option premiums calculated for swaptions with forward maturities of 10 years and expiration dates of 3 months, 1 and 5 years on January 3, 2020 and comparing our values with those of Bloomberg's.

We used Bloomberg's valuations as a proxy for the broad conventional models. Our forward rates were slightly different from Bloomberg's, which could be due to the timing of the data capture or to methodology. This should have minimal effect on the ATM premiums. We also provide a column for the premiums without adjustment to the discount function (See Simozar 2019) for comparison.

Bloomberg's premium for American options are not accurate for any expiration. For example, there is no way to justify a premium of 4.64 for a three months American put while the European put premium is just 1.25. For five-year options, the effect of discount function becomes very significant and none of Bloomberg's premiums are reliable. By comparing the premiums with and without adjustments, we can see that the effect of adjustment is not symmetric, largely due to the lognormal distribution of forward rates, where there is no upper bound for rates. Given the low level of rates, the effect of adjustments is relatively small. During the steep yield curves of 2010-2016, the typical adjustment for the call price was about 10% of its premium.

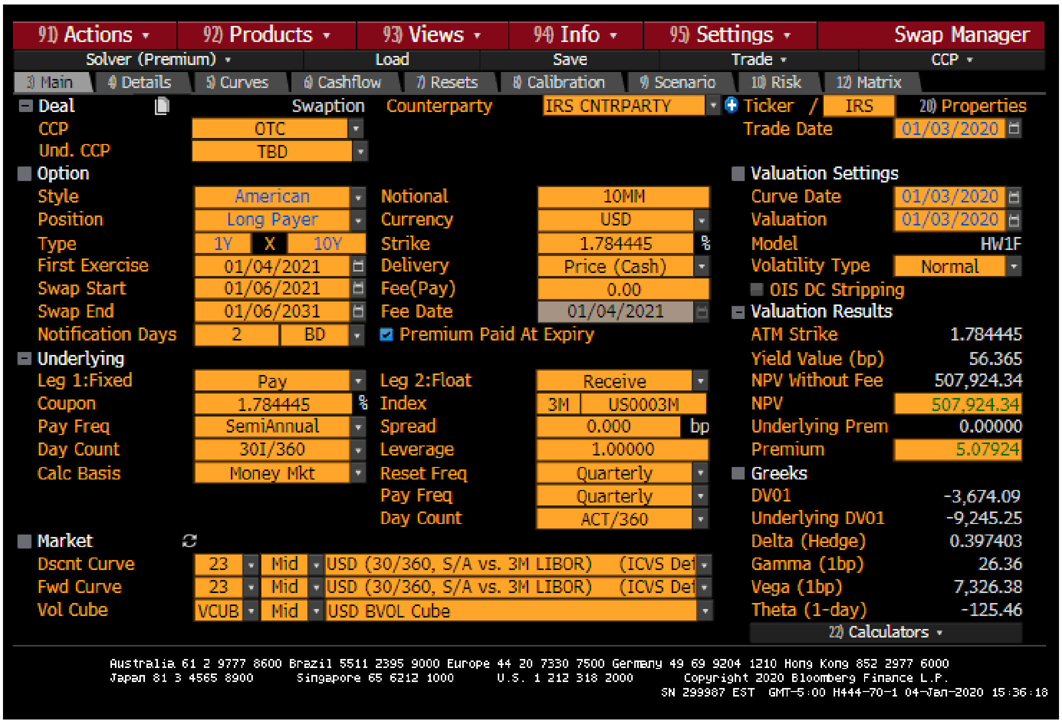

Figure 5. Bloomberg's option premium 1y x10y put ; used with permission of Bloomberg Finance, L.P.

7. Number of Layers

While our calculations for every layer of American options are very accurate, our methodology doesn't account for the value of premium for interlayer exercise of the options. As such, the calculated premium is understated. Another way to look at the exercise boundary is the point that the marginal time value of the option is equal to the marginal accrual of interest rate difference between the underlying and the then prevailing rates. Thus, for a call, just below the exercise boundary, the option is already in the money, but not deep enough for exercise. For example, if we use 20 layers for a 5-year option we would be monitoring the option once a quarter. Using the volatilities and rates in Table 1, we can estimate a deviation of $3.14 \times 23.6/\sqrt{8} = 25.9\,bps$. The expected accrual for this rate change for the second half of the quarter will be $25.9/8 = 3.2\,bps$. This is an underestimate of how much a continuously monitored option will be worth.

The primary contribution to the premium for interlayer exercise is from interlayer volatility which is proportional to the square root of interlayer spacing, which itself is inversely proportional to the number of layers.

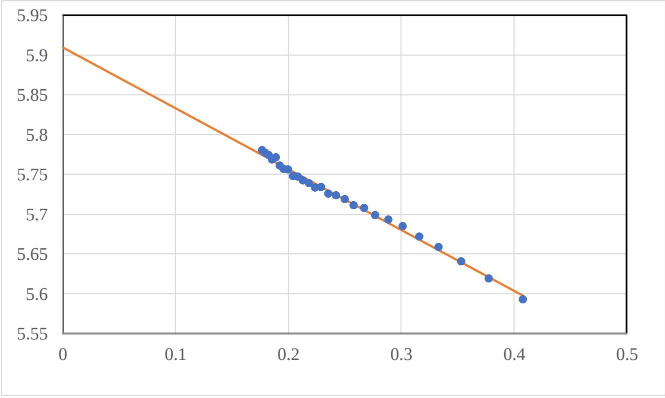

Figure 6. Option premium as a function of $1/\sqrt{\text{Number of Layers}}$

Figure 6 is a plot of the calculated premium of a 5-year call swaption in Table 1 as a function of the square root of the inverse of the number of steps. We used 16 layers for calculating premiums in Table 1. Using market volatility and considering that the interlayer volatility is not constant and the fact that a bond at a given layer is different from a bond in the next layer, the linear fit of the premiums is very good with an R^2 of 0.994. We can estimate the true premium by extrapolation to be about 5.91. If we use just two points on the curve, e.g., 8 and 16 layers and extrapolate the data we will get a premium of 5.91 and if we use 8 and 24 points on the curve, we will get a premium of 5.90.

Similar to binomial option pricing models the convergence rate of the premium is proportional to $\sqrt{N}$. However, there is a relatively large difference between our model and binomial tree models. Most of the interlayer contribution to our model comes from early exercise, where the exercise boundary is deep in the money and the premium gained for early exercise is very large. For binomial trees, the tree is very sparsely populated at the early stages of the tree and such gains cannot be easily realized.

8. Remarks on Bond Options

In Table 1 we saw that the conventional models overestimate the premiums of American options for risk free bonds. Granted that the methods presented in this paper incur a much higher CPU cost, the accuracy that it offers outweighs the extra computational time. Even though there are no closed form solutions for bond options, we can achieve comparable accuracy by using interpolation as in Figure 6. However, the interpolation gain is only 2-3%, while the conventional methods appear to be off by as much as 50-100%. Risk free European options, which are by far the easiest to analyze are not immune to corrections; the adjustment resulting from the discount function for the call and put options results in a price divergence of about 10% for a 5-year option.

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Appendix

In the bucket approach, we convert lognormal distribution of rates to a unit normal distribution (UND) as follows:

$$\begin{aligned} \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{\left(\ell n(y/y_0)-\mu\right)^2}{2\sigma_y^2}} d\ell n\left(y\right) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\ell n(y/y_0)-\mu\right)^2}{2\sigma_y^2}} \frac{d\ell n\left(y\right)}{\sigma_y} \\ &\equiv \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \\ y &= y_0 e^{u\sigma_y+\mu} \end{aligned} \tag{A.1}$$

The UND is divided into 80 intervals or buckets (81 points) covering -6σ to +6σ. For the first layer, the probability distribution is known analytically from the sigma and drift of the first layer. We take out all the buckets that are exercisable on this first layer and evolve the remaining buckets to the next layer, one bucket at a time. The probability distribution of the remaining layers, can theoretically be calculated analytically, however, it will become computationally prohibitive with multiple layers. However, the total weight of each bucket can be calculated very accurately for each layer and then it can be used to evolve the distribution to the following layers.

In order to calculate the weight distribution of a bucket, we take the bucket plus its two adjacent neighbors and we fit their weights to a quadratic equation. If we label the buckets as 1, 2 and 3 and assume that the center of the middle bucket is at the origin, with a width of Δx , the function *z(x)* represents the weight distribution of the buckets. Using a Taylor expansion, we can write,

$$
\begin{aligned}
& z = z_0 + \left.\frac{\partial z}{\partial x}\right|_0 x + \frac{1}{2}\left.\frac{\partial^2 z}{\partial x^2}\right|_0 x^2 = a_1 + m_1 x + \frac{1}{2} c_1 x^2 \\
& \int_{-\frac{3}{2}\Delta x}^{-\frac{1}{2}\Delta x} z dx = w1 \\
& \int_{-\frac{1}{2}\Delta x}^{\frac{1}{2}\Delta x} z dx = w2 \\
& \int_{\frac{1}{2}\Delta x}^{\frac{3}{2}\Delta x} z dx = w3
\end{aligned} \tag{A.2}
$$

The parameters *a*, *m* and *c*, representing level, slope and curvature can be calculated as follows

$$
\begin{aligned}
c_1 &= \frac{w_1 + w_3 - 2w_2}{\Delta x^3} \\
m_1 &= \frac{w_3 - w_1}{2\Delta x^2} \\
a_1 &= \frac{w_2 - \frac{c}{24}\Delta x^3}{\Delta x}
\end{aligned} \tag{A.3}
$$

For points on the tail of the distribution, quadratic approximation does not work since the weights can be exponentially decaying. For example, if the range of weights is a factor of 100, then quadratic fit can result in some points having a negative weight which is not possible. For such buckets, the log of weights can be fit to a quadratic equation. This can be of significance at the exercise boundary where only part of a bucket is used. If the weights are decreasing or increasing almost exponentially, we can use the following method.

$$
\begin{aligned}
q_i &= \ell n(w_i) \\
c_q &= \frac{q_1 + q_3 - 2q_2}{\Delta x^3} \\
m_q &= \frac{q_3 - q_1}{2\Delta x^2} \\
a_q &= \frac{q_2 - \frac{c}{24}\Delta x^3}{\Delta x}
\end{aligned} \tag{A.4}
$$

The resulting quadratic equation represents the log of weight of a bucket not the function of that weight. Therefore, it is not integrable. In order to calculate the area of half a bucket in the interval ($x, x+\Delta x/2$), we start from a point that the density is negligible, e.g., at $x+10\Delta x$ and work backward as follows.

$$\begin{aligned}
&\ell n(w_i) = a_q + m_q x_i + \frac{1}{2} c_q x_i^2 = f(x_i) \\
&x_{1,10} = x + 10\Delta x \\
&x_{2,10} = x + 10.5\Delta x \\
&w_{1,10} = \exp(f(x_{1,10})) \\
&w_{2,10} = \exp(f(x_{2,10})) \\
&w = \sum_{i=10}^{0} (w_{1,10} - w_{2,10})
\end{aligned} \tag{A.5}$$

Given, the quadratic fit to the weight distribution, there are two sets of corrections that need to be made, Intralayer price correction and Interlayer weight correction. The intralayer correction is used for price calculation by making adjustments to the calculated price of the option, given that the bucket approach assumes that the weight and price for each bucket is concentrated in its center.

The interlayer correction is used to adjust for the weight of an unexercised bucket as it evolves to the next layer, considering that its weight is not centered at its middle.

Intralayer Correction – We assume that the price distribution, like the weight distribution is quadratic and write it as

$$\begin{aligned}
&p(x) = p_0 + \left.\frac{\partial p}{\partial x}\right|_0 x + \frac{1}{2} \left.\frac{\partial^2 p}{\partial x^2}\right|_0 x^2 = a_2 + m_2 x + \frac{1}{2} c_2 x^2 \\
&p(-\Delta x) = p_1 = a_2 - m_2 \Delta x + \frac{1}{2} c_2 \Delta x^2 \\
&p(0) = p_2 = a_2 \\
&p(\Delta x) = p_3 = a_2 + m_2 \Delta x + \frac{1}{2} c_2 \Delta x^2
\end{aligned} \tag{A.6}$$

Leading to

$$\begin{aligned}
&c_2 = \frac{w_1 + w_3 - 2w_2}{\Delta x^2} \\
&m_2 = \frac{w_3 - w_1}{2\Delta x} \\
&a_2 = p_2
\end{aligned} \tag{A.7}$$

The contribution of a bucket to the price of the option without correction is simply its weight multiplied by its price, i.e.,

$$\Delta p = p_2 w_2 = p_2 \int_{x_1}^{x_2} (a_1 + m_1 x + \frac{c_1}{2} x^2) dx \tag{A.8}$$

The contribution to the price including the adjustment to the second order will be

$$\Delta p + adj = \int_{x_1}^{x_2} (a_1 + m_1 x + \frac{c_1}{2} x^2)(a_2 + m_2 x + \frac{c_2}{2} x^2) dx \tag{A.9}$$

After simplification and calculating the integral, we find

$$adj = \frac{a_1 m_2}{2} (x_2^2 - x_1^2) + \frac{a_1 c_2 + 2 m_1 m_2}{6} (x_2^3 - x_1^3) \tag{A.10}$$

Except for the bucket at the exercise boundary, for all other buckets $x_1 = -\Delta x$ and $x_2 = \Delta x$. Thus, the adjustment can be simplified to

$$adj = \frac{a_1 c_2 + 2 m_1 m_2}{24} \Delta x^3 \tag{A.11}$$

Interlayer Correction – In our bucket approach, we assume that the weight of a bucket and the price of the option or the forward bond in the bucket are concentrated in the middle of the bucket. However, given the shape of a normal or lognormal distribution, the center of gravity of a bucket is not in the middle, but is slightly closer to the middle of the distribution than the tails. This will result in a slight flattening of the distribution when all the points in a layer are evolved to the next layer. The buckets that are within $\pm\sigma$ lose weight and other buckets gain weight due to the change in the curvature of the distribution at $\pm\sigma$. After about 25 layers the bucket in the middle of the distribution loses about 1% of its weight, which is very small, but can have a measurable effect on the price of an option. Making adjustments to the weights, resolves this issue.

The buckets in each layer are structured from its UND, but their breadth is governed by their overall sigma. When evolving a bucket from one layer to the next, we have to consider the fact that they have different scaling factors, equal to their sigma.

Now, consider a bucket in layer *n-1* which needs to be distributed to a bucket in the next layer *n*, centered at point *b*, using an interlayer sigma of σ_i. The distribution density at point *b* (measured relative to the middle of bucket in layer *n-1*) will be

$$\psi = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{b^2}{2\sigma_i^2}} \tag{A.12}$$

The distribution density at point *b* for a point that is at *x*, relative to the center of the bucket will be

$$\psi = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{(b-x)^2}{2\sigma_i^2}} \tag{A.13}$$

Thus, the total contribution to a bucket in layer *n* centered at *b* with a size of Δu will be

$$\Delta w_{n,b} = \frac{1}{\sqrt{2\pi}\sigma_i} \int_{-\frac{\Delta x}{2}}^{-\frac{\Delta x}{2}} (a_1 + m_1 x + w_1 x^2) dx \int_{b-\frac{\Delta u}{2}}^{b+\frac{\Delta u}{2}} e^{-\frac{(u-x)^2}{2\sigma_i^2}} du \tag{A.14}$$

Using a Taylor series expansion of the probability distribution around *b* and simplifying the result and noting that the contribution without the adjustment will simply be,

$$\Delta w_{n,b} = \frac{1}{\sqrt{2\pi}\sigma_i} \int_{-\frac{\Delta x}{2}}^{-\frac{\Delta x}{2}} (a_1 + m_1 x + w_1 x^2) dx \int_{b-\frac{\Delta u}{2}}^{b+\frac{\Delta u}{2}} e^{-\frac{u^2}{2\sigma_i^2}} du \tag{A.15}$$

We can calculate the adjustment to the distribution considering that its weight is not all concentrated at its middle, will simplify to

$$\Delta w\ adj = \int_{x_1}^{x_2} (a_1 + m_1 x + \frac{c_1}{2} x^2)(-m_3 x + \frac{c_3}{2} x^2) \Delta u dx$$

$$m_3 = \left.\frac{\partial \psi}{\partial u}\right|_b = \frac{-be^{-\frac{b^2}{2\sigma_i^2}}}{\sqrt{2\pi}\sigma_i^3} \tag{A.16}$$

$$c_3 = \left.\frac{\partial^2 \psi}{\partial u^2}\right|_b = \frac{(b^2 - \sigma_i^2)e^{-\frac{b^2}{2\sigma_i^2}}}{\sqrt{2\pi}\sigma_i^5}$$

Thus, the interlayer adjustment can be written as

$$\Delta w\ adj = -\frac{a_1 m_3}{2}(x_2^2 - x_1^2)\Delta u + \frac{a_1 c_3 - 2m_1 m_3}{6}(x_2^3 - x_1^3)\Delta u \tag{A.17}$$

For buckets that are not on the exercise boundary, this is simplified to

$$\Delta w\ adj = \frac{a_1 c_3 - 2m_1 m_3}{24} \Delta x^3 \tag{A.18}$$

Note the sign difference between (A.10) and (A.17)